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Foliation-Preserving Maps Between Solvmanifolds

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Abstract. For $i = 1, 2$, let Γ_i be a lattice in a simply connected, solvable Lie group G_i, and let X_i be a connected Lie subgroup of G_i. The double cosets $\Gamma_i g X_i$ provide a foliation $\mathcal{F}_i$ of the homogeneous space $\Gamma_i\backslash G_i$. Let f be a continuous map from $\Gamma_1\backslash G_1$ to $\Gamma_2\backslash G_2$ whose restriction to each leaf of $\mathcal{F}_1$ is a covering map onto a leaf of $\mathcal{F}_2$. If we assume that $\mathcal{F}_1$ has a dense leaf, and make certain technical assumptions on the lattices Γ_1 and Γ_2, then we show that f must be a composition of maps of two basic types: a homeomorphism of $\Gamma_1\backslash G_1$ that takes each leaf of $\mathcal{F}_1$ to itself, and a map that results from twisting an affine map by a homomorphism into a compact group. We also prove a similar result for many cases where G_1 and G_2 are neither solvable nor semisimple.

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1. Introduction

Let Γ_1 be a lattice in a simply connected, solvable Lie group G_1. Any connected Lie subgroup X_1 of G_1 acts by translations on the homogeneous space $\Gamma_1\backslash G_1$; the orbits of this action are the leaves of a foliation $\mathcal{F}_1$ of $\Gamma_1\backslash G_1$. We call this *the foliation of* $\Gamma_1\backslash G_1$ *by cosets of* X_1. Now suppose Γ_2 is a lattice in some other simply connected, solvable Lie group G_2, and that X_2 is a connected Lie subgroup of G_2, with corresponding foliation $\mathcal{F}_2$ of $\Gamma_2\backslash G_2$. It is natural to ask whether $\mathcal{F}_1$ is topologically equivalent to $\mathcal{F}_2$, or, more generally, whether there is a continuous map f from $\Gamma_1\backslash G_1$ to $\Gamma_2\backslash G_2$ whose restriction to each leaf of $\mathcal{F}_1$ is a covering map onto a leaf of $\mathcal{F}_2$. If so, it is of interest to know all the possible maps f.

Under the assumption that some leaf of $\mathcal{F}_1$ is dense, and technical assumptions on the lattices Γ_1 and Γ_2, we show that every possible f is a composition of maps of the basic types described in Example 1.1 below. (Remarks 2.1 and 2.2 show that there are always finite covers of $\Gamma_1\backslash G_1$ and $\Gamma_2\backslash G_2$ that satisfy the technical assumptions on the lattices.) The reader may note that the composition of maps of the types described in 1.1B and 1.1C is an affine map; the composition of types 1.1B and 1.1C′ is a *doubly crossed* affine map (cf. [9, Definition. 7.3]).

EXAMPLE 1.1.

(A) If $\mathcal{F}_1 = \mathcal{F}_2$, let f be a homeomorphism of $\Gamma_1\backslash G_1$ that maps each leaf of $\mathcal{F}_1$ to itself.

(B) If $\Gamma_1\backslash G_1 = \Gamma_2\backslash G_2$, and $X_1 = r^{-1}X_2 r$ is conjugate to X_2, let f be a translation: $f(\Gamma_1 g) = \Gamma_1 g r^{-1}$.

(C) If there is a continuous group homomorphism $k\colon G_1 \to G_2$ such that $k(\Gamma_1) \subset \Gamma_2$, and the restriction $k|_{X_1}$ of k to X_1 is an homeomorphism onto X_2, let $f\colon \Gamma_1\backslash G_1 \to \Gamma_2\backslash G_2$ be the map induced by k: $f(\Gamma_1 g) = \Gamma_2 k(g)$.

(C′) A map $f\colon \Gamma_1\backslash G_1 \to \Gamma_2\backslash G_2$ of type C can usually be modified as follows. Embed G_2 as a closed subgroup of some solvable Lie group G_2^{T}. For $i = 1, 2$, let T_i be a compact, abelian subgroup of G_2^{T}, and let $\delta_i\colon G_1 \to T_i$ be a homomorphism, such that $\delta_i(\Gamma_1) = e$. Define $\phi\colon G_1 \to G_2^{\mathrm{T}}$ by $\phi(g) = k(g) \cdot \delta_1(g) \cdot \delta_2(g)$. Under appropriate hypotheses (see 2.3), $\phi(X_1)$ is a subgroup of G_2^{T} (even though ϕ is usually not a group homomorphism), and the restriction of ϕ to each coset of X_1 is a homeomorphism onto a coset of $\phi(X_1)$.

Let G_2' be any connected Lie subgroup of G_2^{T} that contains $\phi(G_1)$, and let Γ_2' be a lattice in G_2'. Then the cosets of the subgroup $\phi(X_1)$ provide a foliation $\mathcal{F}_2'$ of $\Gamma_2'\backslash G_2'$. Assume $\Gamma_2 \subset \Gamma_2'$, so ϕ induces a well-defined map $f_2\colon \Gamma_1\backslash G_1 \to \Gamma_2'\backslash G_2'$ defined by $f_2(\Gamma_1 g) = \Gamma_2'\phi(g)$. The restriction of f_2 to each leaf of $\mathcal{F}_1$ is a covering map onto a leaf of $\mathcal{F}_2'$.

One could add more homomorphisms δ_3, δ_4, etc., but Theorem 1.4 shows that this is not necessary.

(C″) Instead of assuming that $\delta_i(\Gamma) = e$, the construction described in C′ can still be carried out if we make the weaker assumption that $\delta_1(\gamma)\delta_2(\gamma) = e$, for all $\gamma \in \Gamma$.

The precise statement of our result requires the definition of the almost-Zariski closure of a subgroup.

DEFINITION 1.2 ([9, Definition 3.2]). A subgroup A of $\mathrm{GL}_n(\mathbb{R})$ is *almost Zariski closed* if there is a Zariski closed subgroup B of $\mathrm{GL}_n(\mathbb{R})$, such that $B^\circ \subset A \subset B$, where B° is the identity component of B in the topology of $\mathrm{GL}_n(\mathbb{R})$ as a C^∞ manifold (not the Zariski topology). There is little difference between being Zariski closed and almost Zariski closed, because B° always has finite index in B.

DEFINITION 1.3 ([9, Definition 3.6]). Let A be a subgroup of $\mathrm{GL}_n(\mathbb{R})$. The *almost-Zariski closure* $\bar{A}$ of A is the unique smallest almost-Zariski closed subgroup that contains A. In particular, if A is a subgroup of a Lie group G, we use $\overline{\mathrm{Ad}_G A}$ to denote the almost-Zariski closure of $\mathrm{Ad}_G A$ in $\mathrm{GL}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of G.

MAIN THEOREM 1.4. *Let X_1 and X_2 be connected Lie subgroups of simply connected, solvable Lie groups G_1 and G_2, respectively. For $i = 1, 2$, let Γ_i be a lattice in G_i. Assume that $\overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad}\,G_1}$, and that $\overline{\mathrm{Ad}_{G_2}\Gamma'}$ is connected, for every subgroup Γ' of Γ_2. Assume, furthermore, that the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 has a dense leaf.*

Let f be a continuous map from $\Gamma_1\backslash G_1$ to $\Gamma_2\backslash G_2$, such that the restriction of f to each leaf of the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 is a covering map onto a leaf of the foliation of $\Gamma_2\backslash G_2$ by cosets of X_2. Then there exists

- *a map $a: \Gamma_1\backslash G_1 \to \Gamma_1\backslash G_1$ of type 1.1A;*
- *a map $b: \Gamma_2\backslash G_2 \to \Gamma_2\backslash G_2$ of type 1.1B; and*
- *a map $c: \Gamma_1\backslash G_1 \to \Gamma_2\backslash G_2$ of type 1.1C″,*

such that $f(\Gamma_1 g) = b(c(a(\Gamma g)))$, for all $g \in G_1$.

In the definition of c, we may take G_2^{T} as defined in Definition 1.5. We may take T_1 to be the elliptic part of $k(X_1)$ and let T_2 be an appropriately chosen elliptic part of $r^{-1}X_2 r$, where k is the homomorphism used in the construction of c, and r is the element of G_2 used in the construction of b.

DEFINITION 1.5 (cf. Proposition 2.4). Let G be a simply connected, solvable Lie group, and let T_G be a maximal compact torus of $\overline{\mathrm{Ad}\, G}$. Define $G^{\mathsf{T}} = G \rtimes T_G$.

For any connected Lie subgroup X of G^{T}, there is a compact, Abelian subgroup T_X of G^{T}, such that $\mathrm{Ad}_G T_X$ is a maximal compact torus of $\overline{\mathrm{Ad}_G X}$. The subgroup T_X is the *elliptic part* of X; it is unique up to conjugation by an element of X.

The *nonelliptic part* of X is the unique simply connected Lie subgroup Y of G^{T} such that $XT_X = YT_X$ and $\overline{\mathrm{Ad}_G Y}$ has no nontrivial compact subgroup.

The theorem was proved by D. Benardete [1, Theorem A(b)] in the special case where X_1 and X_2 are one-dimensional, the map f is a homeomorphism, and the almost-Zariski closures $\overline{\mathrm{Ad}\, G_1}$ and $\overline{\mathrm{Ad}\, G_2}$ have no nontrivial compact subgroups. (However, he proved only that *some* foliation-preserving homeomorphism is a composition of the standard types, not that *all* are.) D. Witte [7, Theorem 5.1] removed the dimension restriction on the subgroups X_1 and X_2, and replaced it with the weaker hypothesis that they are unimodular. We use the same methods as Benardete and Witte. The map δ^* does not appear in the conclusions of [1] and [7], because T_1 and T_2 must be trivial if $\overline{\mathrm{Ad}\, G_2}$ has no compact subgroups.

If the foliation of $\Gamma_1\backslash G_1$ is not assumed to have a dense leaf, then it is not possible to obtain such a precise global conclusion about the form of f. However, the proof shows that there is a homomorphism $k: G_1 \to G_2^{\mathsf{T}}$ with $k(\Gamma_1) \subset \Gamma_2$, such that $k(X_1)$ and $r^{-1}X_2 r$ have the same nonelliptic part, for some $r \in G_2$.

D. Benardete and S. G. Dani [2] recently provided families of examples G, Γ, and X_1, such that $\overline{\mathrm{Ad}_G \Gamma} \neq \overline{\mathrm{Ad}\, G}$, yet, if the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 is topologically equivalent to the foliation of $\Gamma_1\backslash G_1$ by cosets of X_2, then X_1 is conjugate to X_2. The foliations are topologically equivalent to linear foliations of ordinary tori (by applying Remark 2.2), but not via affine maps.

The previous work of D. Benardete [1] and D. Witte [7] requires G_1 and G_2 to be either solvable or semisimple. This is because the proofs rely on the Mostow Rigidity Theorem, which, until recently, was only known in these cases. Now that results of this type have been generalized to other groups [9, § 9], the proof can be generalized. Therefore, in the final section of this paper, we sketch an application of Benardete's method to many groups that are neither solvable nor semisimple. However, unlike our work in the solvable case, our results in this general setting are not at all definitive, because we impose severe restrictions on the subgroups X_1 and X_2. (However, the restrictions are automatically satisfied if X_1 and X_2 are one-dimensional.) We have not attempted to push these methods to their limit, because it seems clear that new ideas will be needed to settle the general case.

2. Preliminaries

2.1. TECHNICAL ASSUMPTIONS ON THE LATTICES

The following remarks show that the assumptions on the lattices Γ_1 and Γ_2 in the statement of Theorem 1.4 can be satisfied by passing to finite covers of $\Gamma_1\backslash G_1$ and $\Gamma_2\backslash G_2$. Therefore, modulo finite covers, the theorem describes the foliation-preserving maps for the natural foliations of all solvmanifolds.

Remark 2.1. The assumption in Theorem 1.4 that $\overline{\mathrm{Ad}_{G_2}\Gamma'}$ is connected, for every subgroup Γ' of Γ_2, can always be satisfied by replacing Γ_2 with a finite-index subgroup (cf. [5, Theorem 6.11, p. 93]), or, in other words, by passing to a finite cover of $\Gamma_2\backslash G_2$. (This may also require Γ_1 to be replaced by a finite-index subgroup, so that the map f is still well-defined.) However, the proof of the theorem does not require the full strength of even this mild assumption. Rather, there is one particular subgroup Γ' whose almost-Zariski closure needs to be connected; see the first paragraph of the proof of the theorem. In particular, if f is a homeomorphism, then we need only assume $\overline{\mathrm{Ad}_{G_2}\Gamma_2}$ is connected.

Remark 2.2. The assumption in Theorem 1.4 that $\overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad}\, G_1}$ is restrictive, but it does not limit the applicability of the result very severely, because the theorem applies to a certain natural finite cover $\Gamma\backslash G$ of $\Gamma_1\backslash G_1$, which we now describe. (Note, however, that the covering map is usually not affine; G is *not* isomorphic to G_1.) Because X_1 has a dense orbit on $\Gamma_1\backslash G_1$, it is easy to see that $\overline{\mathrm{Ad}_{G_1} X_1}$ contains a compact torus T of $\overline{\mathrm{Ad}\, G_1}$, such that $\overline{\mathrm{Ad}_{G_1}\Gamma_1}T = \overline{\mathrm{Ad}\, G_1}$. Therefore, the nilshadow construction (cf. [9, Proposition 8.2]) yields a simply connected, normal subgroup G of $G_1 \rtimes T$, such that

- G contains a finite-index subgroup Γ of Γ_1,
- $\overline{\mathrm{Ad}_G\Gamma} = \overline{\mathrm{Ad}\, G}$, and
- $GT = G_1T$.

Define $\Delta\colon G_1 \to G$ by $\Delta(g) \in gT$. Then Δ is a homeomorphism, and $\Delta(\gamma g) = \gamma\Delta(g)$, for all $\gamma \bullet \Gamma$ and $g \bullet G$. Therefore, Δ^{-1} induces a finite-to-one covering map $\Delta^*\colon \Gamma\backslash G \to \Gamma_1\backslash G_1$. Furthermore, because T normalizes X_1, we see that, letting $X = \Delta(X_1)$, we have $\Delta(gX_1) = \Delta(g)X$, so X is a subgroup of G, and Δ^* maps each leaf of the foliation of $\Gamma\backslash G$ by cosets of X to a leaf of the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1.

2.2. HYPOTHESES NEEDED FOR EXAMPLES OF TYPE 1.1(C″)

The following lemma describes some simple hypotheses that guarantee the conditions needed for the construction of examples of type 1.1(C′) or 1.1 (C″).

LEMMA 2.3. *For $i = 1, 2$, let G_i be a simply connected, solvable Lie group, let T_i be a compact, Abelian subgroup of G_2^{T}, and let $\delta_i\colon G_1 \to T_i$ be a continuous group homomorphism. Let $k\colon G_1 \to G_2^{\mathsf{T}}$ be a continuous group homomorphism, and let X_1 be a connected closed subgroup of G_1, such that $k|_{X_1}$ is a homeomorphism onto $k(X_1)$. Define $\phi_i\colon G_1 \to G_2^{\mathsf{T}}$ by*

$$\phi_1(g) = k(g) \cdot \delta_1(g) \quad \textit{and} \quad \phi_2(g) = k(g) \cdot \delta_1(g) \cdot \delta_2(g).$$

If

$$[k(G_1), T_1] \subset k(\ker \delta_1 \cap \ker \delta_2)$$

and

$$[k(X_1), T_1] \subset k(X_1 \cap \ker \delta_1 \cap \ker \delta_2),$$

then $\phi_1(G_1)$ and $\phi_1(X_1)$ are subgroups of G_2^{T}, and the restriction of ϕ_1 to each left coset of X_1 is a homeomorphism onto a left coset of $\phi_1(X_1)$. If, furthermore,

$$T_2 \textit{ normalizes } \phi_1(X_1),$$

then $\phi_2(X_1)$ is a subgroup, and the restriction of ϕ_2 to each left coset of X_1 is a homeomorphism onto a left coset of $\phi_2(X_1)$.

Proof. We give here only the last part of the proof, showing that the restriction of ϕ_2 to each coset of X_1 is a homeomorphism onto a coset of $\phi_2(X_1)$, because the rest is very similar. Given $g \in G_1$ and $x \in X_1$, we have

$$\begin{aligned}
\phi_2(gx) &= k(gx)\delta_1(gx)\delta_2(gx) \\
&= k(g)k(x)\delta_1(g)\delta_1(x)\delta_2(g)\delta_2(x) && (k, \delta_1, \delta_2 \text{are homomorphisms}) \\
&= k(g)\delta_1(g)k(xy)\delta_1(x)\delta_2(g)\delta_2(x) && (\exists y \in X_1 \cap \ker \delta_1 \cap \ker \delta_2) \\
&= k(g)\delta_1(g)k(xy)\delta_1(xy)\delta_2(xy)\delta_2(g) && (y \in \ker \delta_i \text{ and } T_2 \text{ is Abelian}) \\
&= k(g)\delta_1(g)\phi_2(xy)\delta_2(g) \\
&= k(g)\delta_1(g)\delta_2(g)\phi_2(x') && (T_2 \subset N(\phi_1(X_1)) \Rightarrow \exists x' \in X_1) \\
&= \phi_2(g)\phi_2(x').
\end{aligned}$$

So $\phi_2(gX_1) = \phi_2(g)\phi_2(X_1)$.

We now show that $\phi_2|_{gX_1}$ is one-to-one. Suppose $\phi_2(gx_1) = \phi_2(gx_2)$, for some $g \in G_1$ and $x_1, x_2 \in X_1$. Let $x_1', x_2' \in X_1$ with $k(x_j)\delta_1(g) = \delta_1(g)k(x_j')$ and $\delta_i(x_j') = \delta_i(x_j)$ for $i, j = 1, 2$. Then

$$\begin{aligned}
\phi_1(g)\phi_1(x_1')\delta_2(x_1')\delta_2(g) &= k(g)\delta_1(g)k(x_1')\delta_1(x_1')\delta_2(x_1')\delta_2(g) \\
&= k(g)k(x_1)\delta_1(g)\delta_1(x_1)\delta_2(x_1)\delta_2(g) \\
&= k(gx_1)\delta_1(gx_1)\delta_2(gx_1) \\
&= \phi_2(gx_1) \\
&= \phi_2(gx_2) \\
&= \phi_1(g)\phi_1(x_2')\delta_2(x_2')\delta_2(g) \quad \text{(reverse steps)}.
\end{aligned}$$

Therefore $\phi_1(x_1')\delta_2(x_1') = \phi_1(x_2')\delta_2(x_2')$. Because $\phi_1|_{X_1}$ is a homeomorphism onto $\phi_1(X_1)$, we know that $\phi_1(X_1)$ is simply connected, so it has no nontrivial compact subgroups (see Lemma 2.11). Thus, $\phi_1(X_1) \cap T_2 = e$, so because $\phi_1(x_1')\delta_2(x_1') = \phi_1(x_2')\delta_2(x_2')$, we conclude that $\phi_1(x_1') = \phi_1(x_2')$, so $x_1' = x_2'$. In particular, we have $\delta_i(x_1) = \delta_i(x_1') = \delta_i(x_2') = \delta_i(x_2)$ for $i = 1, 2$, so the equation $\phi_2(gx_1) = \phi_2(gx_2)$ immediately implies $k(x_1) = k(x_2)$. Therefore $x_1 = x_2$.

The restriction of k to X_1 is proper (since it is a homeomorphism onto its image), so the restriction of ϕ_2 to gX_1 is proper. From the preceding paragraph, we know that it is also injective. Therefore, it is a homeomorphism onto its image (see Lemma 2.16).

2.3. ELLIPTIC AND NONELLIPTIC PARTS OF A SUBGROUP

The following proposition justifies the assertions in Definition 1.5, and establishes some basic facts that will often be used without specific reference.

PROPOSITION 2.4. *Let G be a simply connected, solvable Lie group, let T_G be a maximal compact torus of $\overline{\operatorname{Ad} G}$, define $G^{\mathsf{T}} = G \rtimes T_G$, and let X be a connected Lie subgroup of G^{T}. Then*

(1) $[G^{\mathsf{T}}, G^{\mathsf{T}}] = [G, G]$,
(2) *$Z(G^{\mathsf{T}})$ has no nontrivial, compact subgroups,*
(3) *there is a compact, Abelian subgroup T_X of G^{T}, unique up to conjugation by an element of X, such that $\operatorname{Ad}_G T_X$ is a maximal compact torus of $\overline{\operatorname{Ad}_G X}$,*
(4) *there is a unique closed, simply connected subgroup Y of G^{T} such that $XT_X = YT_X$ and $\overline{\operatorname{Ad}_G Y}$ has no nontrivial compact subgroup,*
(5) *we have $[XT_X, XT_X] \subset X \cap Y$,*
(6) *$T_X \cap Y = e$, and*
(7) *Y is normal in XT_X.*

Proof. Lemma 2.12 asserts that $[G^{\mathsf{T}}, G^{\mathsf{T}}] = [G, G]$.

Every compact subgroup of G^{T} is conjugate to a subgroup of T_G (see 2.13). Since T_G is a subgroup of $\operatorname{Aut} G$, we know that no nontrivial element of T_G centralizes G. Therefore $Z(G^{\mathsf{T}})$ has no nontrivial, compact subgroups.

All maximal compact tori of $\overline{\operatorname{Ad} G}$ are conjugate under $\operatorname{Ad} G$ (e.g., see [8, Corollary 4.22]), so, replacing T_G by a conjugate, we may assume T_G contains a maximal compact torus S of $\overline{\operatorname{Ad}_G X}$. Then the desired subgroup T_X is simply S, thought of as a subgroup of $T_G \subset G^{\mathsf{T}}$. The uniqueness follows from the fact that all maximal compact tori of $\overline{\operatorname{Ad}_G A}$ are conjugate under $\operatorname{Ad} A$ (e.g., see [8, Corollary 4.22]).

Assume, as in the preceding paragraph, that T_G contains T_X. There is a natural projection from $\overline{\operatorname{Ad} G}$ to T_G, given by the splitting $\overline{\operatorname{Ad} G} = (A \times T_G) \ltimes U$, where A is a maximal $\mathbb{R}$-split torus and U is the unipotent radical. Let $\sigma\colon G^{\mathsf{T}} \to T_G$ be the composite homomorphism

$$\sigma\colon G^{\mathsf{T}} \xrightarrow{\operatorname{Ad}} \overline{\operatorname{Ad} G} \xrightarrow{\text{projection}} T_G \xrightarrow{x \mapsto x^{-1}} T_G\,,$$

and define $\Delta\colon G^{\mathsf{T}} \to G^{\mathsf{T}}$ by $\Delta(g) = g\sigma(g)$, so Δ is a *nilshadow map* [9, Definition 4.1]. Then $\Delta(G^{\mathsf{T}})$ is a subgroup of $\Delta(G^{\mathsf{T}})$ [9, Corollary 4.8]. Since Δ is obviously a proper map, we know that $\Delta(G)$ is closed. Since $T_X \subset T_G$, we have $\sigma(X) \subset T_X$, so $\Delta(X)$ is a subgroup of $\Delta(G)$ [9, Corollary 4.9] and, obviously, $AT_X = \Delta(X)T_X$. By construction, $\overline{\mathrm{Ad}_G\,\Delta(G^{\mathsf{T}})}$ has no nontrivial compact subgroup [9, Proposition 4.10]. Since $Z(G^{\mathsf{T}})$ has no nontrivial, compact subgroups, this implies that $\Delta(G^{\mathsf{T}})$ has no nontrivial, compact subgroups, which means that $\Delta(G^{\mathsf{T}})$ is simply connected (see Lemma 2.11). Therefore, we may let $Y = \Delta(X)$ (see Lemma 2.9). If Y' is any nonelliptic part of X, then, because $\overline{\mathrm{Ad}_G\,Y'}$ has no nontrivial compact subgroup, the subgroup Y' must be contained in the kernel of σ. This kernel is precisely $\Delta(G^{\mathsf{T}})$, so it is not difficult to see that $Y' = Y$.

The definition of the nilshadow map Δ immediately implies $X \cap \ker\sigma \subset \Delta(X)$. Since T_G is Abelian, $\ker\sigma$ must contain $[G^{\mathsf{T}}, G^{\mathsf{T}}]$. Therefore $X \cap [G^{\mathsf{T}}, G^{\mathsf{T}}] \subset Y$. Then, because Lemma 2.12 implies $[XT_X, XT_X] = [X, X] \subset X$, we have $[XT_X, XT_X] \subset X \cap [G^{\mathsf{T}}, G^{\mathsf{T}}] \subset X \cap Y$.

Being simply connected, Y has no nontrivial, compact subgroups (see Lemma 2.11), so $T_X \cap Y = e$.

Since $[Y, XT_X] \subset [XT_X, XT_X] \subset X \cap Y \subset Y$, we see that Y is normal in XT_X.

2.4. NONDIVERGENT SUBGROUPS

DEFINITION 2.5 ([7, § 4]). Let X and Y be subsets of a Lie group G. We say that *X does not diverge from Y* if there is a compact subset K of G with $X \subset YK$. If X does not diverge from Y, and Y does not diverge from X, then we may say that *X and Y do not diverge from each other.*

For the special case where the subgroup Y is unimodular, the following proposition was proved by D. Witte [7, Cors. 4.10 and 4.11].

PROPOSITION 2.6. *Let X and Y be connected Lie subgroups of a simply connected solvable Lie group G, and assume that $\overline{\mathrm{Ad}\,G}$ has no nontrivial compact subgroups. If X does not diverge from Y, then $X \subset Y$.*

Proof. Each element of $[G, G]$ acts unipotently, hence unimodularly, on each subspace of $\mathcal{G}$ that it normalizes, so $[G, G] \cap Y \subset \ker \Delta_Y$, where Δ_Y is the modular function of Y. Hence, Δ_Y extends to a continuous homomorphism $\Delta\colon G \to \mathbb{R}^+$. Define a semidirect product $G \ltimes \mathbb{R}$, by letting each $g \in G$ act on $\mathbb{R}$ via multiplication by $1/\Delta(g)$, so $Y \ltimes \mathbb{R}$ is a unimodular subgroup of $G \ltimes \mathbb{R}$. Since X does not diverge from Y in G, we see that X does not diverge from $Y \ltimes \mathbb{R}$ in $G \ltimes \mathbb{R}$, so the proof of [7, Corollary 4.11] shows that $X \subset Y \ltimes \mathbb{R}$. (Although the statement of the corollary assumes both X and Y are unimodular, the proof only requires this assumption for Y.) Hence, $X \subset (Y \ltimes \mathbb{R}) \cap G = Y$, as desired.

COROLLARY 2.7. *Let G be a simply connected, solvable Lie group, and let X and Y be connected Lie subgroups of G^{T}. If X does not diverge from Y, then the nonelliptic part of X is contained in the nonelliptic part of Y.*

Proof. Letting X_1, Y_1 and G_1 be the nonelliptic parts of X, Y, and G, respectively, it is easy to see that X_1 does not diverge from Y_1 in G_1. Therefore, Proposition 2.6 implies $X_1 \subset Y_1$, as desired.

EXAMPLE 2.8. Let $G = \widetilde{\mathrm{SO}}(2) \ltimes \mathbb{R}^2$, where $\widetilde{\mathrm{SO}}(2)$ is the universal cover of SO(2), and let X be a subgroup of G that is conjugate to $\widetilde{\mathrm{SO}}(2)$. Then X does not diverge from $\widetilde{\mathrm{SO}}(2)$ in $G^{\mathrm{T}} \cong \mathbb{R} \times (\mathrm{SO}(2) \rtimes \mathbb{R}^2)$, but X need not be contained in $\widetilde{\mathrm{SO}}(2)$, so we see that the conclusion of the corollary cannot be strengthened to say that $X \subset Y$.

2.5. MISCELLANEOUS FACTS

For ease of reference, we record some basic results on solvable groups.

LEMMA 2.9 ([3, Theorem XII.2.2, p. 137]). *Every connected subgroup of a simply connected, solvable Lie group G is closed and simply connected.*

LEMMA 2.10 ([9, Lemma 2.17]). *Let N be a closed subgroup of a connected, solvable Lie group G. Then G/N is simply connected if and only if N is connected and contains a maximal compact subgroup of G.*

LEMMA 2.11 ([9, Corollary 2.18]). *A connected, solvable Lie group is simply connected if and only if it has no nontrivial compact subgroups.*

LEMMA 2.12 (cf. [9, Lemma 3.24]). *If G is a connected, solvable Lie group, and T is an Abelian subgroup of $\overline{\mathrm{Ad}\, G}$, then $[G \rtimes T, G \rtimes T] = [G, G] \rtimes e$.*

PROPOSITION 2.13 ([3, Theorem XV.3.1, pp. 180–181]). *Every compact subgroup of a connected Lie group G is contained in a maximal compact subgroup, and all maximal compact subgroups of G are conjugate.*

LEMMA 2.14. *Let Γ_1 be a lattice in a simply connected, solvable Lie group G_1, such that $\overline{\mathrm{Ad}_{G_1} \Gamma_1} = \overline{\mathrm{Ad}\, G_1}$, and let E be a compact, Abelian Lie group. Let X_1 be a connected Lie subgroup of G_1, and assume that the foliation of $\Gamma_1 1$ by cosets of X_1 has a dense leaf. Suppose $\tau\colon X_1 \to E$ and $\delta^*\colon \Gamma_1 1 \to E$ are continuous maps, such that $\delta^*(\Gamma_1) = e$, and $\delta^*(px) = \delta^*(p)\tau(x)$, for every $p \bullet \Gamma_1 1$ and $x \in X_1$. Then there is a continuous homomorphism $\widetilde{\delta}\colon G_1 \to E$ such that $\delta^*(\Gamma_1 g) = \widetilde{\delta}(g)$ for every $g \in G_1$.*

Proof. (cf. [1, Proof on p. 502]). For $x, y \in X_1$, we have

$$\delta^*(\Gamma_1)\tau(xy) = \delta^*(\Gamma_1 xy) = \delta^*(\Gamma_1 x)\tau(y) = \delta^*(\Gamma_1)\tau(x)\tau(y),$$

so we see that τ is a homomorphism. Because X_1 is simply connected (see 2.9), we may lift τ to a homomorphism $\widetilde{\tau}\colon X_1 \to \widetilde{E}$, where $\widetilde{E}$ is the universal cover of E.

Because the foliation of $\Gamma_1\backslash G_1$ has a dense leaf, we may assume $\Gamma_1 X_1$ is dense in G_1. Because G_1 is simply connected, we may lift δ^* to a map $\widetilde{\delta}\colon G_1 \to \widetilde{E}$ with $\widetilde{\delta}(e) = e$. The restriction of $\widetilde{\delta}$ to the fundamental group Γ_1 is a homomorphism into $\widetilde{E}$. Theorem 2.15 implies that this restriction $\widetilde{\delta}|_{\Gamma_1}$ extends to a continuous homomorphism $k\colon G_1 \to \widetilde{E}$.

We have $\widetilde{\delta}(\gamma g) = k(\gamma) \cdot \widetilde{\delta}(g)$, for every $\gamma \in \Gamma_1$ and $g \in G_1$. Therefore, because $\Gamma_1\backslash G_1$ is compact, there is a compact subset K of $\widetilde{E}$, such that $\widetilde{\delta}(g) \in k(g)K$, for all $g \in G_1$. In particular, for $x \in X_1$, we have $\widetilde{\tau}(x) = \widetilde{\delta}(x) \in k(x)K$, so the difference $\widetilde{\tau} - k$ is a homomorphism with bounded image. Therefore, Lemma 2.11 implies that the image is trivial, which means $\widetilde{\tau} = k$, so $\widetilde{\delta}$ agrees with k on X_1. Since they also agree on Γ_1, and $\Gamma_1 X_1$ is dense in G_1, this implies that $\widetilde{\delta} = k$ is a homomorphism.

In the statement of the following result in [9], it is assumed that the maximal compact torus T_{G_2} used in the construction of G_2^{T} contains a maximal compact torus of $\overline{\mathrm{Ad} G_2 \Gamma_1^{\alpha}}$. Because all maximal compact tori of $\overline{\mathrm{Ad}\ G_2}$ are conjugate under Ad G_2, this assumption is unnecessary.

THEOREM 2.15 ([9, Corollary 6.5]). *Let Γ_1 be a lattice in a simply connected, solvable Lie group G_1, and assume $\overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad} G_1}$. Let G_2 be a simply connected, solvable Lie group. If α is a homomorphism from Γ_1 into G_2, such that $\overline{\mathrm{Ad}_{G_2}\Gamma_1^{\alpha}}$ is connected, then α extends to a continuous homomorphism from G_1 to G_2^{T}.*

For convenience, we also note the following well-known, simple lemma.

LEMMA 2.16. *Every continuous, proper bijection between locally compact Hausdorff topological spaces is a homeomorphism.*

3. Proof of the Main Theorem

The outline of this proof is based on [7, § 6]. However, complications are caused by the possible lack of an inverse to f, and by the possible existence of nontrivial compact subgroups of $\overline{\mathrm{Ad}\, G_2}$.

Proof of Theorem 1.4. By composing f with the translation by some element $r \in G_2$, we may assume without loss of generality that $f(\Gamma_1) = \Gamma_2$. Then, because G_1 is simply connected, we may lift f to a map $\widetilde{f}\colon G_1 \to G_2$ with $\widetilde{f}(e) = e$. Because Γ_i is the fundamental group of $\Gamma_i\backslash G_i$, we see that the restriction of $\widetilde{f}$ to Γ_1 is a homomorphism into Γ_2. Because $\mathrm{Ad}_{G_2}\widetilde{f}(\Gamma_1)$ is connected, Theorem 2.15 implies that this restriction $\widetilde{f}|_{\Gamma_1}$ extends to a continuous homomorphism $k\colon G_1 \to G_2^{\mathrm{T}}$.

Remark 3.1. We have $\widetilde{f}(\gamma g) = k(\gamma) \cdot \widetilde{f}(g)$, for every $\gamma \in \Gamma_1$ and $g \in G_1$. Therefore, because $\Gamma_1\backslash G_1$ is compact, there is a compact subset K of G_2^{T}, such that $\widetilde{f}(g) \in k(g)K$, for all $g \in G_1$. Hence, for every subset A of G, the sets $\widetilde{f}(A)$ and $k(A)$ do not diverge from each other (see Definition 2.5).

Step 1. *The restriction of $\tilde{f}$ to each coset of X_1 is a homeomorphism onto a coset of X_2.* By assumption, the restriction of f to each leaf of the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 is a covering map onto a leaf of the foliation of $\Gamma_2\backslash G_2$ by cosets of X_2. Therefore, the restriction of f to each coset of X_1 is a covering map onto a coset of X_2. Because X_1 and X_2 are simply connected (see Lemma 2.9), this covering map must be a homeomorphism.

Step 2 [7, *Step* 3 *of Proof of Theorem* 6.1]. *$k(X_1)$ and X_2 have the same nonelliptic part; call it Y.* Because k maps X_1 to $k(X_1)$, and $\tilde{f}$ maps X_1 to X_2, Remark 3.1 implies that $k(X_1)$ and X_2 do not diverge from each other. Therefore, Corollary 2.7 implies that $k(X_1)$ and X_2 have the same nonelliptic part.

Step 3. *The restriction of k to X_1 is a homeomorphism onto $k(X_1)$, and $k(X_1)$ is closed.* Since $\tilde{f}|_{X_1}$ is a homeomorphism onto X_2, it is a proper map. Therefore, Remark 3.1 implies that $k|_{X_1}$ is also a proper map. This implies $k(X_1)$ is closed. It also implies that the kernel of $k|_{X_1}$ is compact. Then Lemma 2.11 implies that the kernel is trivial, so $k|_{X_1}$ is injective. Thus, $k|_{X_1}$ is an isomorphism onto its image [6, Lemma 2.5.3, p. 59].

Step 4 [7, *Step* 4 *of Proof of Theorem* 6.1]. *For $g \in G_1$, define $\delta(g) \in G_2^{\mathrm{T}}$ by: $\tilde{f}(g) = k(g) \cdot \delta(g)$; then $\delta(g)$ normalizes Y, and $\delta(\gamma g) = \delta(g)$, for every $\gamma \bullet \Gamma_1$, so δ factors through to a well-defined map $\delta^*\colon \Gamma_1\backslash G_1 \to N_{G_2^{\mathrm{T}}}(Y)$.* Because $\tilde{f}$ maps cosets of X_1 to cosets of X_2, we have

$$k(g)^{-1}\tilde{f}(gX_1) = k(g)^{-1}\tilde{f}(g)X_2 = \delta(g)X_2 = \big(\delta(g)X_2\delta(g)^{-1}\big) \cdot \delta(g).$$

Then, because Remark 3.1 implies that $k(g)^{-1}\tilde{f}(gX_1)$ and $k(X_1) = k(g)^{-1}k(gX_1)$ do not diverge from each other, this implies that the subgroups $\delta(g)X_2\delta(g)^{-1}$ and $k(X_1)$ do not diverge from each other. Therefore, Corollary 2.7 implies that the nonelliptic part of $\delta(g)X_2\delta(g)^{-1}$ is the same as the nonelliptic part of $k(X_1)$, namely, Y. On the other hand, because the nonelliptic part of X_2 is Y (see Step 2), it is obvious that the nonelliptic part of $\delta(g)X_2\delta(g)^{-1}$ is $\delta(g)Y\delta(g)^{-1}$. Therefore, $Y = \delta(g)Y\delta(g)^{-1}$.

Because $\tilde{f}(\gamma g) = k(\gamma) \cdot \tilde{f}(g)$, it is easy to see that $\delta(\gamma g) = \delta(g)$.

Step 5 [7, *Step* 5 *of proof of Theorem* 6.1]. *Let T_1 and T_2 be the elliptic parts of $k(X_1)$ and X_2, respectively; then $\delta(g) \in T_1T_2Y$, for every $g \in G_1$.* For $x_1 \in X_1$, we have $\tilde{f}(gx_1) = \tilde{f}(g) \cdot x_2$ for some $x_2 \in X_2$, so

$$k(g)k(x_1)\delta(gx_1) = k(gx_1) \cdot \delta(gx_1) = \tilde{f}(gx_1) = \tilde{f}(g)x_2 = k(g) \cdot \delta(g) \cdot x_2.$$

Writing $k(x_1) = t_1y_1$ and $x_2 = t_2y_2$ for some $t_1 \bullet T_1$, $t_2 \bullet T_2$, and $y_1, y_2 \in Y$, we then have $t_1y_1 \cdot \delta(gx_1) = \delta(g) \cdot t_2y_2$. This implies that the map $\overline{\delta}\colon \Gamma_1\backslash G_1 \to T_1\backslash N_{G_2^{\mathrm{T}}}(Y)/T_2Y$, induced by δ, is constant on each leaf of the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1. Because this foliation has a dense leaf, this implies that $\overline{\delta}$ is constant. Because $\tilde{f}(e) = e = k(e)$, we know that $\delta(e) = e$, so this implies that $\delta(g)$ belongs to T_1T_2Y for every $g \in G_1$, as desired.

Step 6. *We may assume* $(T_1T_2) \cap Y = e$; *then the maps*

$$\begin{aligned} &\nu\colon Y \times T_1T_2 \to YT_1T_2\colon && (y,t) \mapsto ptyt, \\ &\nu_1\colon X_1 \times T_1T_2 \to k(X_1)T_1T_2\colon && (x,t) \mapsto k(x)t, \quad \text{and} \\ &\nu_2\colon X_2 \times T_2T_1 \to X_2T_2T_1\colon && (x,t) \mapsto xt \end{aligned}$$

are homeomorphisms. Let $S = T_1 \cap (T_2Y)$. Then there is some $g \in T_2Y$, such that $S \subset g^{-1}T_2g$ (see 2.13). We have

$$T_1 \cap \big((g^{-1}T_2g)Y\big) = T_1 \cap (T_2Y) = S \subset g^{-1}T_2g,$$

so, by replacing the choice T_2 of the elliptic part of X_2 with the equally valid choice $g^{-1}T_2g$, we may assume $T_1 \cap (T_2Y) \subset T_2$. We now show that this implies $(T_1T_2) \cap Y = e$: if $t_1t_2 \in Y$, with $t_1 \in T_1$ and $t_2 \in T_2$, then $t_1 \in T_1 \cap (T_2Y) \subset T_2$, so $t_1t_2 \in T_2 \cap Y = e$, as desired.

The maps ν, ν_1, and ν_2 are obviously continuous, surjective, and proper. (For the properness of ν_1, recall that $k|_{X_1}$ is a homeomorphism onto $k(X_1)$.) Thus, it suffices to show that they are injective (see 2.16).

Suppose $y't'_1t'_2 = yt_1t_2$, for some $y', y \in Y$ and $t'_i, t_i \in T_i$. Then $t_1^{-1}t'_1t'_2t_2^{-1} \in (T_1T_2) \cap Y = e$, so $t'_1t'_2 = t_1t_2$ and, hence, $y' = y$, so ν is injective.

Suppose $k(x')t'_1t'_2 = k(x)t_1t_2$, for some $x', x \in X_1$ and $t'_i, t_i \in T_i$. We have $k(x') = y't'$ and $k(x) = yt$, for some $y', y \in Y$ and $t', t \in T_1$. Then $y't't'_1t'_2 = ytt_1t_2$, so, from the conclusion of the preceding paragraph, we must have $y' = y$. Since $k(X_1) \cap T_1 = e$, this implies $k(x') = k(x)$. Therefore $x = x'$, so ν_1 is injective. A similar argument applies to ν_2.

Warning 3.2. The compact set T_1T_2 need not be a subgroup of G_2^{T}, because T_1 and T_2 need not commute with each other.

Step 7. *There is a left* Γ_1*-equivariant homeomorphism* ϕ *of* G_1, *such that* $\tilde{f}(g) \in k(\phi(g))T_1T_2$, *for every* $g \in G_1$, *and* ϕ *takes each left coset of* X_1 *onto itself.* Since $\delta(g) \in T_1T_2Y = k(X_1)T_1T_2$, there is a unique element $\chi(g)$ of X_1, such that $\delta(g) \in k(\chi(g))T_1T_2$; namely, $\chi(g)$ is the first coordinate of $\nu_1^{-1}(\delta(g))$, so χ is a continuous function of g. Define $\phi(g) = g \cdot \chi(g)$, so $\phi\colon G_1 \to G_1$ is continuous, and takes each left coset gX_1 of X_1 into itself. Then

$$\tilde{f}(g) = k(g)\delta(g) \in k(g)k(\chi(g))T_1T_2 = k(g \cdot \chi(g))T_1T_2 = k(\phi(g))T_1T_2. \tag{3.3}$$

So all that remains is to show that ϕ is left Γ_1-equivariant and has a continuous inverse.

Note that, because $\delta(\gamma g) = \delta(g)$, we must have $\chi(\gamma g) = \chi(g)$, for all $g \in G_1$ and $\gamma \in \Gamma_1$. Therefore

$$\phi(\gamma g) = (\gamma g) \cdot \chi(\gamma g) = \gamma(g \cdot \chi(g)) = \gamma\phi(g),$$

which is exactly what it means to say that ϕ is left Γ_1-equivariant.

Define

$$\zeta\colon G_1 \times X_1 \times T_2T_1 \to G_1 \times (X_2T_2T_1) \text{ by } \zeta(g,x,t) = (g, \tilde{f}(g)^{-1}\tilde{f}(gx)t).$$

Similarly, we have $[k(G_1), T_1] \subset k(\ker \delta_1 \cap \ker \delta_2)$.

For any $x \in X_1$, we have

$$\phi_1(x)\delta_2(x) = k(x)\delta_1(x)\delta_2(x) = \tilde{f}(x) \in X_2 \subset YT_2$$

and $\delta_2(x) \in T_2$, so $\phi_1(X_1) \subset YT_2$. Therefore, from Step 9, we have

$$[\phi_1(X_1), T_2] \subset [YT_2, T_2] = [Y, T_2] \subset (Y \cap X_2)^\circ.$$

Define a map $\xi\colon X_1 \to Y$ by $\xi(x) = k(x)\tau_1(x)^{-1}$. Then ξ is bijective, because $k(X_1)T_1 = YT_1$ and $k(X_1) \cap T_1 = e$. It is also proper, because $k|_{X_1}$ is proper and T_1 is compact. Therefore ξ is a homeomorphism (see 2.16).

Let $y \in (Y \cap X_2)^\circ$. Since $y \in Y$, there is some $x \in X_1$, such that $\xi(x) = y$. Since $y \in X_2$, we have $\tau_2(x) = e$, so $x \in \ker \tau_2$. Because ξ is a homeomorphism, we know $\xi^{-1}((Y \cap X_2)^\circ)$ is connected, so we conclude that $x \in (\ker \tau_2)^\circ \subset \ker \delta_2$ (see Step 11). Also, from Step 10, we have $\tau_1(x)^{-1}\tau_2(x) = \delta(x)$. Therefore

$$\tau_1(x)^{-1} = \tau_1(x)^{-1} \cdot e = \tau_1(x)^{-1}\tau_2(x) = \delta(x) = \delta_1(x)\delta_2(x) = \delta_1(x) \cdot e = \delta_1(x),$$

so

$$y = k(x)\tau_1(x)^{-1} = k(x)\delta_1(x) = \phi_1(x) \in \phi_1(X_1).$$

Therefore $[\phi_1(X_1), T_2] \subset \phi_1(X_1)$, which means T_2 normalizes $\phi_1(X_1)$.

4. Non-Solvable Groups

DEFINITION 4.1 ([9, Definition 9.1]). A lattice Γ in a connected Lie group G is *superrigid* if, for every homomorphism $\alpha\colon \Gamma \to \mathrm{GL}_n(\mathbb{R})$, such that $\overline{\Gamma^\alpha}$ has no non-trivial, connected, compact, semisimple, normal subgroups, there is a continuous homomorphism $\beta\colon G \to \overline{\Gamma^\alpha}$, such that β agrees with α on a finite-index subgroup of Γ.

Remark 4.2. In the context of Definition 4.1, suppose G_2 is any connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ that contains Γ^α. Then β induces a homomorphism $\bar{\beta}\colon G_2 \to \overline{G_2}/G_2$. Since $\overline{G_2}/G_2$ is abelian, and $\bar{\beta}$ is trivial on a finite-index subgroup of the lattice Γ, we see that $\bar{\beta}(G)$ is compact and Abelian. Therefore $\beta(G) \subset G_2S$, for any maximal compact torus S of $\overline{\mathrm{Rad}\, G_2}$. Therefore, letting $G_2^T = G_2 \rtimes T$, for any maximal compact torus T of $\overline{\mathrm{Ad}_{G_2} \mathrm{Rad}\, G_2}$, we see that there is a homomorphism $k\colon G \to G_2 \rtimes T$, such that k agrees with α on a finite-index subgroup of Γ.

DEFINITION 4.3. Let us say that a connected Lie group G is *almost linear* if there is a continuous homomorphism $\beta\colon\ G \to \mathrm{GL}_n(\mathbb{R})$, for some n, such that the kernel of β is finite.

The following two theorems combine to show that many lattices are superrigid. Furthermore, by considering induced representations, it is easy to see that every finite-index subgroup of a superrigid lattice is superrigid.

THEOREM 4.4 (Margulis [4, Theorem IX.5.12(ii), p. 327]). *Let G be a simply connected, almost linear, semisimple Lie group, such that* $\mathbb{R}$-rank$(L) \geqslant 2$, *for every simple factor L of G. Then every lattice Γ in G is superrigid.*

THEOREM 4.5 ([9, Theorem 9.9]). *Let Γ be a lattice in a simply connected, almost linear Lie group G, and assume that G has no nontrivial, connected, compact, semisimple, normal subgroups. Then Γ is superrigid if*

- $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}\, G}$; *and*
- *the image of Γ in $G/\mathrm{Rad}\, G$ is a superrigid lattice.*

THEOREM 4.6. *For $i = 1, 2$, let X_i be a closed, unimodular subgroup of a simply connected, almost linear Lie group G_i. Assume* $\mathrm{Rad}\, X_i$ *is simply connected, and that X_i has no nontrivial, compact, semisimple quotients. Also assume there is no connected, closed, normal subgroup N of $[X_1, X_1]$, such that $\overline{\mathrm{Ad}_{[X_1,X_1]/N} X_1}$ is compact and nontrivial.*

For $i = 1, 2$, let Γ_i be a lattice in G_i. Assume that G_i has no nontrivial, connected, compact, semisimple, normal subgroups, that $\overline{\mathrm{Ad}_{G_i}\Gamma_i} = \overline{\mathrm{Ad} G_i}$, and that Γ_1 is superrigid in G_1.

Assume, furthermore, that the foliation of $\Gamma_1'\backslash G_1$ by cosets of X_1 has a dense leaf, for every finite-index subgroup Γ_1' of Γ_1.

Let $f\colon \Gamma_1'\backslash G_1 \to \Gamma_2\backslash G_2$ be a homeomorphism, such that f maps each leaf of the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 onto a leaf of the foliation of $\Gamma_2\backslash G_2$ by cosets of X_2.

Then, for some finite-index subgroup Γ_1' of Γ_1, there exists

- *a map $a\colon \Gamma_1'\backslash G_1 \to \Gamma_1'\backslash G_1$ of type* 1.1*A;*
- *a map $b\colon \Gamma_2\backslash G_2 \to \Gamma_2\backslash G_2$ of type* 1.1*B; and*
- *a map $c\colon \Gamma_1'\backslash G_1 \to \Gamma_2\backslash G_2$ of type* 1.1*C″,*

such that $f(\Gamma_1 g) = b(c(a(\Gamma_1' g)))$, for all $g \in G_1$.

In the definition of c, we may take G_2^{T} as defined in Remark 4.2. *We may take T_1 to be the elliptic part of $k(X_1)$ and let T_2 be an appropriately chosen elliptic part of $r^{-1}X_2 r$, where k is the homomorphism used in the construction of c, and r is the element of G_2 used in the construction of b.*

Sketch of proof. The proof of Theorem 1.4 applies with only minor changes; we point out the substantial differences.

A change is required already in the first paragraph of the proof. Assume $f(\Gamma_1) = \Gamma_2$. Then f lifts to a homeomorphism $\tilde{f}\colon G_1 \to G_2$ with $\tilde{f}(e) = e$. Since G_2 is almost linear, Γ_1 is superrigid, and G_1 is simply connected, it is not difficult to see from Remark 4.2 that there is a finite-index subgroup Γ_1' of Γ_1, such that $\tilde{f}|_{\Gamma_1'}$ extends to a homomorphism $k\colon G_1 \to G_2^{\mathrm{T}}$. For simplicity, replace Γ_1 with Γ_1'.

See [7, Defnitions. 4.3 and 4.8] for the definition of elliptic and nonelliptic parts of a subgroup of G_2^{T}. Note that the assumptions on X_i imply that T_i is a torus, and $X_i \cap T_i$ is finite. Also, X_i has no nontrivial connected, compact, normal subgroups.

Because $\Gamma_1\backslash G_1$ may not be compact, the second sentence of Remark 3.1 may not be valid, so the arguments of Steps 2 and 4 need to be modified, as in Steps 3 and 4 of the proof of [7, Theorem 6.1].

The conclusion of Step 3 should be weakened slightly: instead of being a homeomorphism, the restriction of k to X_1 is a finite-to-one covering map. Similarly, the maps v, v_1, and v_2 of Step 6 are finite-to-one covering maps. For example, to see this in the case of v, let T_1' be a subtorus of T_1, such that $T_1' \cap T_2$ is finite, and $T_1 = T_1'(T_1 \cap T_2)$. Then the group $(Y \rtimes T_1') \times T_2$ acts on G_2 by $(y, t_1, t_2) \cdot g = yt_1gt_2$. The orbit of e under this action is YT_1T_2, and the stabilizer of e is finite. So the map $(y, t_1, t_2) \mapsto yt_1t_2$ is a covering map with finite fibers. The space $Y \times T_1T_2 \approx ((Y \rtimes T_1') \times T_2)/(T_1' \cap T_2)$ is an intermediate covering space, with covering map v.

In the proof of Step 7, $v_1^{-1}(\delta(g))$ may not be a single point, but, because G_1 is simply connected, there is a continuous function $\hat{\chi}: G_1 \to X_1 \times T_1T_2$ with $v_1(\hat{\chi}(g)) = \delta(g)$ (and $\hat{\chi}(e) = e$). Define $\chi(g)$ to be the first component of $\hat{\chi}(g)$. (A similar device is used to define χ'.) Because v_1 is finite-to-one, and the lift $\hat{\chi}$ is determined by its value at any one point, there is a finite-index subgroup Γ of Γ_1, such that $\chi(\gamma g) = \chi(g)$, for all $g \in G_1$ and $\gamma \in \Gamma$. Then, by replacing Γ_1 with Γ, we may assume ϕ is left Γ_1-equivariant.

Still in the proof of Step 7, to see that ζ is a finite-to-one covering map, note that the map ζ_1: $G_1 \times X_1 \to G_1 \times X_2$ defined by $\zeta_1(g, x) = (g, \tilde{f}(g)^{-1}\tilde{f}(gx))$ is a homeomorphism (by the argument in the last paragraph of Step 7), and we have $\zeta(g, x, t) = (\mathrm{Id} \times v_2)(\zeta_1(g, x), t)$.

Also in the proof of Step 6, let us show that ψ is the inverse of ϕ. For all $g \in G_1$, we have

$$\tilde{f}(g) \in k(\phi(g))T_1T_2 \quad \text{and} \quad \tilde{f}(\psi(\phi(g))) \in k(\phi(g))T_1T_2.$$

Therefore, the two maps $\lambda_1, \lambda_2: G_1 \to X_2 \times T_2T_1$ defined by

$$\lambda_1(g) = \left(e, \tilde{f}(g)^{-1}k(\phi(g))\right)$$

and

$$\lambda_2(g) = \left(\tilde{f}(g)^{-1}\tilde{f}(\psi(\phi(g))), \tilde{f}(\psi(\phi(g)))^{-1}k(\phi(g))\right)$$

both satisfy $v_2(\lambda_i(g)) = \tilde{f}(g)^{-1}k(\phi(g))$. Since $\lambda_1(e) = (e, e) = \lambda_2(e)$, and v_2 is a covering map, this implies $\lambda_1 = \lambda_2$. By comparing the first coordinates, and noting that $\tilde{f}$ is a homeomorphism, we conclude that $\psi \circ \phi$ is the identity map. Similarly, $\phi \circ \psi$ is also the identity.

In Definition 3.4, since $X_i \cap T_i$ may be nontrivial, the functions τ_1 and τ_2 may not be well defined. However, there is a finite cover $\widetilde{X_1}$ of X_1, such that τ_i is a well-defined map from $\widetilde{X_1}$ to T_i with $\tau_i(e) = e$.

The conclusion of Step 9 can be established as follows. Because T_2 normalizes Y, we have $[Y, T_2] \subset Y$. Furthermore, because T_2 is Abelian and is in $\overline{\mathrm{Ad}_{G_2} X_2}$, we have $[T_2X_2, T_2X_2] = [X_2, X_2] \subset X_2$. Therefore, $[Y, T_2] \subset Y \cap X_2$. Since $[X_1, X_1] \subset Y$, we have $\mathrm{Ad}_{[X_1,X_1]/[Y,Y]}\, Y = e$, so $\overline{\mathrm{Ad}_{[X_1,X_1]/[Y,Y]}\, X_1} = \mathrm{Ad}_{[X_1,X_1]/[Y,Y]}\, T_1$ is compact. Therefore, by hypothesis, it must be trivial. Since $\mathrm{Ad}_{T_1 X_1/[X_1,X_1]} T_1$ is also trivial, and $\mathrm{Ad}_{X_1} T_1$ is semisimple, this implies $[T_1 X_1, T_1] \subset [Y, Y] \subset X_2$. Therefore, $[Y, T_1] \subset Y \cap X_2$.

The argument of Step 10 shows that, for all $g \in G_1$ and $x \in \widetilde{X_1}$, we have

$$\delta(gx) \in (T_1 \cap Y)\tau_1(x)^{-1}\delta(g)\tau_2(x)(T_2 \cap Y).$$

(This calculation uses the observation that $T_1 \cap Y$ and $T_2 \cap Y$ are contained in $Y \cap X_2$. To see this, note that $T_i \cap Y$ is contained in a maximal compact subgroup K of Y. Because $\mathrm{Rad}\, Y$ is simply connected, we see that K is contained in a Levi subgroup of Y, so $K \subset [Y, Y] \subset Y \cap X_2$.) Since δ, τ_1, and τ_2 are continuous, and $T_i \cap Y$ is finite, this implies the equation in the conclusion of Step 10.

For Step 11, it is necessary to prove a slightly modified version of Lemma 2.14.

In Step 12, although the map $\xi\colon \widetilde{X_1} \to Y$ is not a homeomorphism, we must have $(Y \cap X_2)^\circ \subset \xi((\ker \tau_2)^\circ)$, because ξ is a covering map.

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