CORRECTION TO "ZERO-ENTROPY AFFINE MAPS ON HOMOGENEOUS SPACES"

By DAVE WITTE

Abstract. Proposition 6.4 of the author's paper [*American Journal of Mathematics* **109** (1987), 927–961] is incorrect. This invalid proposition was used in the proof of Corollary 6.5, so we provide a new proof of the latter result.

Professor A. Starkov has pointed out an error in the proof of Proposition 6.4 of the author's paper [*American Journal of Mathematics* **109** (1987), 927–961]. Contrary to the assertion near the end of the first paragraph, it may not be possible to choose T^* to be a subgroup of $\langle g\rangle^*$. (The problem is that $\langle g\rangle^*$ may not contain a maximal torus of $(\operatorname{Rad} G)^*$, because the maximal torus of $\langle g\rangle^*$ may be diagonally embedded in $\mathrm{LEVI}^* \times T^*$.) The proposition cannot be salvaged, so the claim must be retracted.

Fortunately, Proposition 6.4 was used only in the proof of Corollary 6.5, for which we can give a direct proof. As it is no longer a corollary, we now reclassify this as a proposition.

PROPOSITION 6.5. *Suppose g is an ergodic translation on a locally faithful finite-volume homogeneous space $\Gamma\backslash G$ of a Lie group G, and assume $G = \Gamma G^\circ = G^\circ\langle g\rangle$. If g has zero entropy, then, for some nonzero power g^n of g, there is a finite-volume homogenenous space $\Gamma'\backslash G'$ of some Lie group G' whose radical is nilpotent, and a continuous map $\psi\colon \Gamma'\backslash G' \to \Gamma\backslash G$ that is affine for some translation $g' \in G'$ via g^n. Furthermore, every fiber of ψ is finite.*

Proof. Assume for simplicity that G is connected and simply connected. (A remark on the general case follows the proof.) Because g is ergodic, we may assume $\Gamma\langle g\rangle$ is dense in G.

Let G^* be the identity component of the Zariski closure of $\operatorname{Ad} G$ in $\operatorname{Aut}(\mathcal{G})$, let S^* be a maximal compact torus of the Zariski closure of $\operatorname{Ad}_G\langle g\rangle$, and let L^*T^* be a reductive Levi subgroup of G^*, containing S^*. (So L^* is semisimple and T^* is a maximal torus of $\operatorname{Rad} G^*$ that centralizes L^*. From Proposition 6.2, we know that T^* is compact.) The composition of Ad_G and the projection from G^* onto

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$T^*/(L^* \cap T^*)$ is a homomorphism, which, because G is simply connected, can be lifted to a homomorphism $\pi\colon G \to T^*$. Define a map $\phi\colon G \to G \rtimes T^*$ by $x^\phi = (x, x^{-\pi})$, where $x^{-\pi} = (x^{-1})^\pi$. Notice that

$$(xy)^\phi = x^\phi \cdot (y^{x^{-\pi}})^\phi, \tag{1}$$

for all $x, y \in G$.

The definition of ϕ is based on the nilshadow construction of Auslander and Tolimieri [AT] (or see [W, §4]). In particular, $\operatorname{Rad} G^\phi$ is the nilshadow of $\operatorname{Rad} G$, so $\operatorname{Rad} G^\phi$ is nilpotent.

Let $\overline{\Gamma}\backslash\overline{G}$ be the faithful version of $\Gamma^\phi\backslash G^\phi$. More precisely, let $\overline{\Gamma} = \Gamma^\phi/N$ and $\overline{G} = G^\phi/N$, where N is the largest normal subgroup of G^ϕ contained in Γ^ϕ. We know Γ^π is finite (see Proposition 4.20), so, replacing Γ by a finite-index subgroup, we may assume $\Gamma^\pi = e$. This implies that $(\gamma x)^\phi = \gamma^\phi x^\phi$ for all $\gamma \in \Gamma$ and $x \in G$, so ϕ induces a well-defined homeomorphism $\overline{\phi}\colon \Gamma\backslash G \to \overline{\Gamma}\backslash\overline{G}$.

Unfortunately, $\overline{\phi}$ is not affine for g if T^* does not centralize g. We will compensate for the action of T^* by composing with a twisted affine map. Assume for simplicity that $L^* \cap T^* = e$. (Under this assumption, the map $L^* \times T^* \to L^*T^*$ is an isomorphism. In general, it is a finite cover.) Let $S_L^* = (L^* \cap S^*)^\circ$, and let S_Δ^* be a subtorus of S^* that is complementary to S_L^* and contains $S^* \cap T^*$. Let $S_\perp^*$ be the image of S_Δ^* under the projection $L^* \times T^* \to L^*$, and notice that $S_L^* \cap S_\perp^* = e$. Thus, we have

$$S^* = S_L^* \times S_\Delta^* \subset S_L^* \times S_\perp^* \times T^* \subset L^* \times T^*.$$

Because $\Gamma\langle g\rangle$ is dense in G and $\Gamma^\pi = e$, we see that S_Δ^* finitely covers T^*, via the projection $S_\perp^* \times T^* \to T^*$.

Let L be a semisimple Levi subgroup of G with $\operatorname{Ad}_G L = L^*$, and let $\overline{L}$ be the corresponding Levi subgroup of $\overline{G}$. Since $\operatorname{Rad}\overline{G}$ is nilpotent, and $\overline{\Gamma}\backslash\overline{G}$ is faithful, we know that $Z(\overline{G})$ is compact (see Corollary 4.6), so some compact torus $\overline{S}_\perp \subset \overline{L}$ finitely covers $S_\perp^*$, via the map $\operatorname{Ad}_{\overline{G}}$.

By construction, we know that T^* is finitely covered by S_Δ^*, so the homomorphism $\pi\colon G \to T^*$ lifts to a homomorphism $G \to S_\Delta^*$. By composing this with the projection $S_\Delta^* \to S_\perp^*$, we obtain a homomorphism $G \to S_\perp^*$. Then, since $\overline{S}_\perp$ finitely covers $S_\perp^*$, this lifts to a homomorphism $\overline{\pi}\colon G \to \overline{S}_\perp$. Note that, from the definition of $\overline{\pi}$, we have $(\operatorname{Ad}_{\overline{G}}(x^{\overline{\pi}}), x^\pi) \in S_\Delta^*$, for all $x \in G$. Because $S_\Delta^* \subset S^*$ is contained in the identity component of the Zariski closure of $\operatorname{Ad}_G\langle g\rangle$, which centralizes g, this implies that

$$g^{x^{-\pi}x^{-\overline{\pi}}} = g. \tag{2}$$

Replacing Γ by a finite-index subgroup, we may assume $\Gamma^{\overline{\pi}} = e$; hence $\overline{\pi}$ induces a well-defined map from $\Gamma\backslash G$ to $\overline{S}_\perp$. Thus, we may define a homeomorphism

$$\psi\colon \Gamma\backslash G \to \overline{\Gamma}\backslash\overline{G}\colon\ x \mapsto x^{\overline{\phi}} \cdot x^{-\overline{\pi}}. \tag{3}$$

Then

$$(xg)^{\psi} \overset{(3)}{=} (xg)^{\overline{\phi}} \cdot (xg)^{-\overline{\pi}} \overset{(1)}{=} (x^{\overline{\phi}} g^{x^{-\pi}\phi})(x^{-\overline{\pi}} g^{-\overline{\pi}}) \overset{(3)}{=} x^{\psi} g^{x^{-\pi} x^{-\overline{\pi}}\phi} g^{-\overline{\pi}} \overset{(2)}{=} x^{\psi} g^{\phi} g^{-\overline{\pi}}.$$

In other words, ψ is affine for g via $g^{\phi} g^{-\overline{\pi}}$. □

Remark. If G is not connected, then, because $G = G^{\circ}\langle g\rangle$, there is no harm in assuming $G = G^{\circ} \rtimes \langle g\rangle$, and we may assume G° is simply connected.

Let G^* and G^- be the identity components of the Zariski closures of $\operatorname{Ad} G$ and $\operatorname{Ad} G^{\circ}$, respectively. By replacing g with a power g^n, we may assume $\operatorname{Ad} g \in G^*$. Let S^+ be a maximal compact torus of the Zariski closure of $\operatorname{Ad}_G\langle g\rangle$, and let $S^* = (S^+ \cap G^-)^{\circ}$. Let L^*T^+ be a reductive Levi subgroup of G^*, containing S^+, and define $T^* = T^+ \cap G^-$.

Let $T_L T^+$ be a maximal compact torus in G^*, containing S^+ (where T_L is a compact torus in L^*), so $T_L T^*$ is a maximal compact torus in G^-. Then $T_L T^* S^+ = T_L T^+$, so there is a subtorus Z of S^+ that is almost complementary to $T_L T^*$ in $T_L T^+$. Assume for simplicity that $Z T_L \cap T^* = e$, so there is a natural projection $G^* \to T^*$. Then we may define a homomorphism $\pi\colon G \to T^*$ by composing Ad_G with this projection.

We now construct a semidirect product $G^* \rtimes T^*$. Let

$$H = \frac{G^{\circ} \rtimes G^*}{\{(x^{-1}, \operatorname{Ad} x) \mid x \in [G, G]\}}.$$

We may assume $\operatorname{Ad} g^n \notin \operatorname{Ad} G^{\circ}$, for all $n \neq 0$, for otherwise we could assume G is connected. Therefore, $G \cong G^{\circ}\langle \operatorname{Ad} g\rangle$ injects into H. Because

$$[\operatorname{Ad} g, T^*] \subset [G^*, G^*] = \operatorname{Ad}[G, G],$$

we see that $T^* \subset N_H(G)$. Therefore, T^* is a group of automorphisms of G, so we may form the semidirect product $G \rtimes T^*$.

With these definitions of G^*, T^*, S^*, L^*, π, and $G^* \rtimes T^*$ in hand, we may proceed essentially as in the proof above.

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