Colour-permuting automorphisms of complete Cayley graphs

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Abstract. A bijection f of a metric space is *distance-permuting* if the distance from $f(x)$ to $f(y)$ depends only on the distance from x to y.

For example, it it is known that every distance-permuting bijection of the real line is the composition of an isometry and a dilation ($x \mapsto kx$). So they are affine maps.

We study the analogue in which G is any (finite or infinite) group, and the "distance" from x to y is the "absolute value" of the unique element s of G, such that $xs = y$. We determine precisely which groups have the property that every distance-preserving bijection is an affine map. The smallest exception is the quaternion group of order 8, and all other exceptions are constructed from this one.

It is natural to state the problem in the language of graph-theory: construct a graph by joining each pair of points (x, y) with an edge, and label (or "colour") this edge with its length. Then we are interested in bijections that permute the colours of the edges: i.e., the colour of the edge from $f(x)$ to $f(y)$ depends only on the colour of the edge from x to y.

This is joint work with Shirin Alimirzaei.

Definition

A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ (or any metric space) is **distance-preserving** if

$$d(\varphi(x), \varphi(y)) = d(x, y).$$

Obvious examples are translations, rotations, and reflections.
It is well known that all are a combination of these.

So every distance-pres 0-map (i.e., fixes 0) is **linear**.

$$(\varphi(\vec{x}) = A\vec{x})$$

Remark (Beckman-Quarles Theorem [1953])

If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ preserves distance 1 and $n > 1$,

$$(d(\vec{x}, \vec{y}) = 1 \Rightarrow d(\varphi(\vec{x}), \varphi(\vec{y})) = 1)$$

then φ preserves all distances. (So φ is linear if it fixes 0.)

Every distance-preserving 0-map on $\mathbb{R}^n$ is linear $(A\vec{x})$.

Today: Finite versions of this — and a related problem.

Example

For $x, y \in \mathbb{Z}_n$ (integers mod n), there is a natural distance:

Draw points of $\mathbb{Z}_n$ on a circle.

Distance is how many steps apart.

$$d(x, y) \approx |x - y| \pmod{n}$$

Then every distance-preserving 0-map is linear $(ax,\, a = \pm 1)$.

Proof. $\varphi(1) = \pm 1$, because $\varphi(0) = 0$.
Assume $\varphi(1) = 1$ (compose with $x \mapsto -x$ if necessary).
Show by induction that $\varphi(x) = x$:

$$\varphi(x+1) = x \pm 1 \text{ and } \varphi(x+1) \neq \varphi(x-1) = x - 1.$$

Eg. Every distance-preserving 0-map of $\mathbb{Z}_n$ is linear.

Graph-theoretic interpretation

For $x, y \in \mathbb{Z}_n$:

- Let $d(x,y) = |x - y|$ and $|s| = \{s, -s\}$.
- Draw an edge from x to y and ~~label~~ colour it with $d(x,y)$.

Every colour-pres 0-automorphism of this graph is linear.

Generalization to any group G

For $g, h \in G$:

- Let $d(g,h) = |gh^{-1}|$ and $|s| = \{s, s^{-1}\}$. $(d(g, sg) = |s|)$
- Draw edge from g to h and colour it with $d(g,h)$.

This is the **complete Cayley graph K_G** of G.

¿Is every colour-preserving **1**-automorphism of this graph ~~linear~~ a group automorphism?

Theorem (Byrne-Donner-Sibley, Dobson-Hujdurović-Kutnar-J.Morris)

Every colour-preserving $\mathbf{1}$*-automorphism of* K_G *is a group automorphism unless* $G = Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

Notation (Hamilton's quaternions)

- $i^2 = j^2 = k^2 = -1 \quad ij = k = -ji.$
- $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a nonabelian group of order 8

Example

Define $\varphi(x) = x^{-1}$ on Q_8 (or $Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$).
Then φ is *not* a group automorphism (because Q_8 not abelian).
However $\varphi(ij) = (ij)^{-1} = k^{-1} = -k = i(-j) = i\,\varphi(j)$.
In fact, for all g, s, we have

$$\varphi(sg) = (sg)^{-1} = g^{-1}s^{-1} \stackrel{*}{=} s^{\pm 1}g^{-1} = s^{\pm 1}\varphi(g).$$

So $d(\varphi(g), \varphi(sg)) = d(\varphi(g), s^{\pm 1}\varphi(g)) = |s^{\pm 1}| = d(g, sg)$,
so φ is a colour-preserving.

Shirin and I worked on a generalization of this.

Definition. A 0-map φ is **distance-permuting** if

$$d(\varphi(x), \varphi(y)) \text{ depends only on } d(x, y).$$

I.e., there is a function $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, such that

$$d(\varphi(x), \varphi(y)) = f(d(x, y)).$$

Exercise

Every distance-permuting 0*-map on* $\mathbb{R}^n$ *is linear.*

Proof when $n > 1$.

Compose with a dilation ($x \mapsto ax$) to assume

φ preserves distance 1.

Beckman-Quarles Theorem: φ preserves all distances. □

¿How about the finite (combinatorial) version of this?

Asked by Andrew Fiori.

Theorem (Alimirzaei-Morris [2025])

*Every colour-permuting $\mathbf{1}$-automorphism of K_G is the composition of a group aut and a colour-**preserving** $\mathbf{1}$-aut.*

Recall (Byrne-Donner-Sibley, Dobson-Hujdurović-Kutnar-J.Morris)

Every colour-preserving $\mathbf{1}$-automorphism of K_G is a group automorphism unless $G = Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

Corollary (Alimirzaei-Morris [2025])

Every colour-permuting $\mathbf{1}$-automorphism of K_G is a group aut unless $G = Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

Actually, we derive the "theorem" from the "corollary."
Our direct proof of the corollary relies on lots of previous work on related (harder) problems theorems, including the work on colour-preserving automorphisms.

Here is an example of how results on colour-preserving auts can be used to obtain information on colour-permuting auts.

Recall (Byrne-Donner-Sibley, Dobson-Hujdurović-Kutnar-J.Morris)

Every colour-preserving **1**-automorphism of K_G is a group automorphism unless $G = Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

Corollary (combined with Watkins [1971 Thm. 3])

*If K_G has a nontrivial colour-preserving **1**-automorphism then G is has an abelian subgroup of index ≤ 2 (and G is known).*

Proposition (Alimirzaei-Morris [2025])

*Assume K_G has no nontrival colour-pres **1**-automorphism. Then every colour-permuting **1**-aut of K_G is a group aut.*

So Shirin and I only need to consider a very few groups. We generalized the proposition and used known results.

Proof of the proposition.

Fix some $g, h \in G$, and define the following function, which is the composition of certain translations with φ and its inverse:

$$\psi(x) = \varphi^{-1}(\varphi(x \cdot g^{-1}) \cdot \varphi(g^{-1})^{-1} \cdot \varphi(h)) \cdot h^{-1}.$$

All of the the constituents of ψ are colour-permuting so ψ is colour permuting.

Also, $\psi\colon \mathbf{1} \mapsto g^{-1} \mapsto \varphi(g^{-1}) \mapsto \mathbf{1} \mapsto \varphi(h) \mapsto h \mapsto \mathbf{1}$.

However, translations are colour-preserving, and we applied both φ and φ^{-1} (which cancel ??? each other), so ψ is colour-preserving. By assumption, then ψ is trivial: $\psi(x) = x$.

Unwinding this, we conclude that

$$\varphi(x \cdot h) \cdot \varphi(h)^{-1} = \varphi(x \cdot g^{-1}) \cdot \varphi(g^{-1})^{-1}$$

Letting $x = g$, this implies $\varphi(g \cdot h) \cdot \varphi(h)^{-1} = \varphi(g^{-1})^{-1}$, so

$$\varphi(g \cdot h) = \varphi(g^{-1})^{-1}\varphi(h).$$

Since this is true for all g and h, we conclude that φ respects the group operation (i.e., it is a group homomorphism). Being a bijection, it is therefore a group automorphism. □

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