Hamiltonian paths in cartesian powers of directed cycles

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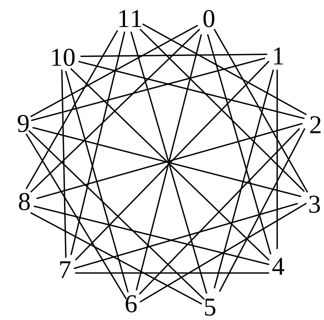
Abstract

The vertex set of the k^{th} cartesian power of a directed cycle of length m can be naturally identified with the abelian group $(\mathbb{Z}_m)^k$. For any two elements $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ of $(\mathbb{Z}_m)^k$, it is easy to see that if there is a hamiltonian path from u to v, then
$$u_1 + \cdots + u_k \equiv v_1 + \cdots + v_k + 1 \pmod{m}.$$
We prove the converse, unless $k = 2$ and m is odd. This is joint work with David Austin and Heather Gavlas. A similar result is conjectured for cartesian products of directed cycles that are not assumed to be of equal lengths.

Notation. Circulant graph $\mathrm{Circ}(n; S)$:

- vertex set $= \mathbb{Z}_n$ (integers modulo n)
- edge v — $v \pm s$ for $s \in S$

Eg. $\mathrm{Circ}(12; 3, 4, 6)$



Exer. Every (connected) circulant graph has a hamiltonian cycle.

Thm (Chen-Quimpo).
Circulant grfs of deg ≥ 3 are hamiltonian conn'd (unless they are bipartite — then laceable).
I.e. $\forall$ *vertices* u, v, $\exists$ *ham path from* u *to* v.

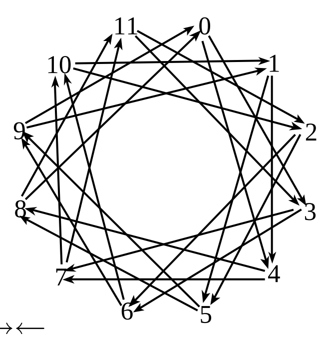
Similar result for circulant **di**graphs?

Notation. Circulant digraph $\overrightarrow{\mathrm{Circ}}(n; S)$:

- vertex set $= \mathbb{Z}_n$ (integers modulo n)
- edge $v \to v + s$ for $s \in S$

Rem. $d^+(\overrightarrow{X}) = 1 \Rightarrow \overrightarrow{X}$ is a directed cycle
$\Rightarrow \overrightarrow{X}$ is hamiltonian.

Eg. $\overrightarrow{\mathrm{Circ}}(12; 3, 4)$ is **not** hamiltonian.



Proof. Spse 0 travels by 4.
$\Rightarrow$ 1 travels by 4.
$\Rightarrow$ 2 travels by 4.
$\Rightarrow \cdots$
Every vertex travels by 4.
$0 \to 4 \to 8 \to 0$ $\quad \to\leftarrow$

Thm (Rankin, 1948). $\overrightarrow{\mathrm{Circ}}(n; a, b)$ *has ham cyc* $\Leftrightarrow$ *number-theoretic condition on* a, b, n.

Now consider $d^+(\overrightarrow{X}) \geq 3$.

Eg. $\overrightarrow{\mathrm{Circ}}(12; 3, 4, 6)$ is **not** hamiltonian.

Thm (Locke-Witte). $\nexists$ *hamiltonian cycle in* $\overrightarrow{\mathrm{Circ}}(12k; 6k-3, 6k-2, 6k)$.

Thm (Locke-Witte). $\nexists$ *hamiltonian cycle in* $\overrightarrow{\mathrm{Circ}}(2k; a, b, b+k) \Leftrightarrow \gcd(a, b, k) = 1$ *and* ...

Rem. 1st thm: antipodal verts are joined by edge.
2nd thm: vert adjacent to 2 antipodal verts.

Rem. $\overrightarrow{\mathrm{Circ}}(n; a, b, c) \cong \overrightarrow{\mathrm{Circ}}(n; xa, xb, xc)$
if $\gcd(x, n) = 1$.

Ques. *Do the two thms (and the remark) give all the nonham, circulant digraphs of outdegree 3?*

Computer search: yes for less than 100 vertices.

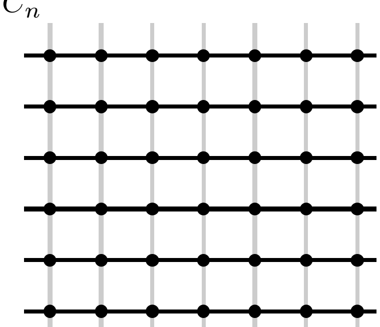
Problem. *Show* $d^+(\overrightarrow{X}) \geq 4 \Rightarrow \overrightarrow{X}$ *has ham cyc.*

Cartesian products of directed cycles.

Notation. Cartesian product $X \times Y$:

- vertex set $V(X) \times V(Y)$
- edge
  - $(x, y) \,\text{—}\, (x', y)$ if $x \,\text{—}\, x'$
  - $(x, y) \,\text{—}\, (x, y')$ if $y \,\text{—}\, y'$.

Eg. $C_m \times C_n$



Exer. Cartesian product of (undirected) cycles is hamiltonian.

Thm (Chen-Quimpo).
$C_{m_1} \times \cdots \times C_{m_r}$ is hamiltonian connected
unless $r = 1$
or each m_i is even (in which case, laceable).

Would like a similar result for the directed case.

First step: show $\overrightarrow{C}_{m_1} \times \cdots \times \overrightarrow{C}_{m_r}$ is hamiltonian.

Thm (Rankin). *$\exists$ ham cyc in $\overrightarrow{C}_m \times \overrightarrow{C}_n$*
$\Leftrightarrow \exists$ rel. prime $s, t \in \mathbb{Z}^+$, $sm + tn = mn$.

Thm (Curran-Witte). *$r > 2 \Rightarrow$ hamiltonian.*

Ques. *Which vertices are joined by a ham path?*

$\overrightarrow{X} = \overrightarrow{C}_{m_1} \times \cdots \times \overrightarrow{C}_{m_r}$

Rem. Identify set of vertices with $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$.
edge $(u_1, \ldots, u_r) \to (v_1, \ldots, v_r)$
$\Rightarrow v_1 + \cdots + v_r = u_1 + \cdots + u_r + 1$
$(\bmod \gcd(m_1, \ldots, m_r))$

$\exists$ path of length ℓ from u to v
$\Rightarrow \sum v_i \equiv \sum u_i + \ell$

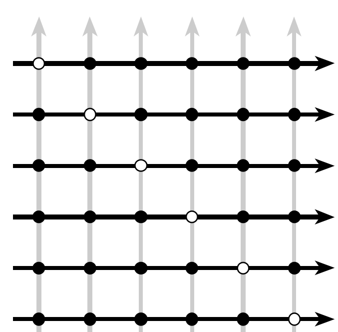
$\exists$ ham path from u to v
$\Rightarrow \sum v_i \equiv \sum u_i + (m_1 m_2 \cdots m_r - 1)$
$\equiv \sum u_i - 1.$

Converse should be true:

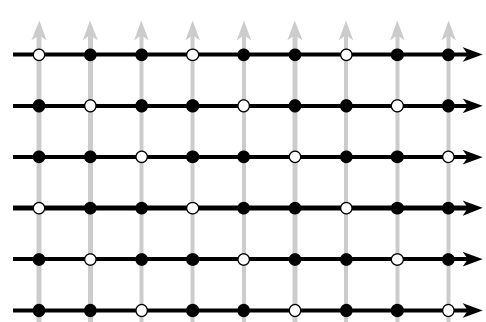
Conj. *Assume $r \geq 3$.*
$\exists$ ham path from u to v
$\Leftrightarrow \sum u_i \equiv 1 + \sum v_i \quad (\bmod \gcd(m_1, \ldots, m_r))$

Rem. If $\gcd(m_1, \ldots, m_r) = 1$ (and $r \geq 3$), then any two vertices should be joined by a ham path.

Eg. $\overrightarrow{C}_m \times \overrightarrow{C}_m$



Eg. $\overrightarrow{C}_9 \times \overrightarrow{C}_6$



Conj. $\exists$ *ham path from u to v*
if $\sum u_i \equiv 1 + \sum v_i \pmod{\gcd(m_1, \ldots, m_r)}$

Thm (Austin-Gavlas-Witte). *Conjecture is true if* $m_1 = m_2 = \cdots = m_r$ *(and $r \geq 3$).*

Proof. Let $m = m_1$ $(= m_2 = \cdots = m_r)$. Assume

- m is even, (!!!)
- $r = 3$, and
- $u = 0$.

Notation. $x = (1,0,0)$, $y = (0,1,0)$, $z = (0,0,1)$.

$D = \{\, \delta \in (\mathbb{Z}_m)^3 \mid \sum \delta_i \equiv 0 \bmod m \,\}$

$\overrightarrow{\Delta} = \overrightarrow{\mathrm{Cay}}(D; y - x, z - x)$.

Note that $\sum v_i \equiv -1 \pmod{m}$, so $v + x \in D$.

Lem. $\exists$ *ham cyc H in $\overrightarrow{\Delta}$, s.t. $d_H(0, v+x)$ is even.*

Prop. $\exists$ *such ham cyc in $\overrightarrow{\Delta}$ $\Rightarrow$ $\exists$ ham path from 0 to v in $(\overrightarrow{C}_m)^3$.*

Cayley digraphs. Generalize circulant graphs to use noncyclic groups:

Defn. $\overrightarrow{\mathrm{Cay}}(G; S)$ for subset S of abelian group G:

- vertex set $= G$
- edge $g \to g + s$ for $s \in S$.

Eg. $\overrightarrow{\mathrm{Circ}}(n; S) = \overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; S)$.

Eg. $(\overrightarrow{C}_m)^3 \cong \overrightarrow{\mathrm{Cay}}\big((\mathbb{Z}_m)^3; x, y, z\big)$.

Prop. $\exists$ *ham cyc H in $\overrightarrow{\Delta}$, with $d_H(0, v+x)$ even $\Rightarrow$ $\exists$ ham path from 0 to v in $(\overrightarrow{C}_m)^3$.*

Proof. Say $d_H(0, v + x) = 2k$.

Let $a = (1, 0)$ and $b = (1, 1)$ in $\mathbb{Z}_m \times \mathbb{Z}_{m^2}$.

$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_m \times \mathbb{Z}_{m^2}; a, b)$ has a hamiltonian path from $(0, 0)$ to $(-1, 2k)$.

Suffices to find $\phi\colon \mathbb{Z}_m \times \mathbb{Z}_{m^2} \to (\mathbb{Z}_m)^3$ such that

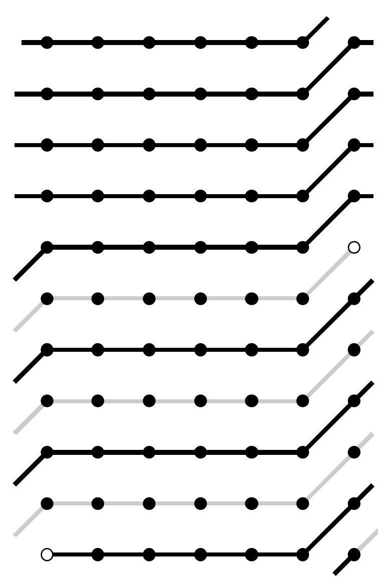
- $\phi(-1, 2k) = v$ and
- ϕ embeds $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_m \times \mathbb{Z}_{m^2}; a, b)$.

Let $H = h_0, \ldots, h_{m^2}$ with $h_0 = h_{m^2} = 0$
so $h_{2k} = v + x$.

Define $\phi(i, j) = ix + h_j$.

- $\phi(-1, 2k) = -x + h_{2k} = v$
- $\phi(v + a) - \phi(v) = x \in \{x, y, z\}$
- $\phi(v + b) - \phi(v) = x + (c_{j+1} - c_j)$
  $\in x + \{y - x, z - x\} = \{y, z\} \subset \{x, y, z\}$.

Ham path $(0, 0) \to (-1, 2k)$
in $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_m \times \mathbb{Z}_{m^2}; a, b)$:



$\big((a^{m-2}, b^2)^k, (a^{m-2}, b, a)^{m^2-2k-1},$
$(a^{m-2}, b^2)^k, a^{m-2}, b\big)$

Notation. $x = (1,0,0)$, $y = (0,1,0)$, $z = (0,0,1)$.

- $D = \{ \delta \in (\mathbb{Z}_m)^3 \mid \sum \delta_i \equiv 0 \bmod m \}$
- $\overrightarrow{\Delta} = \overrightarrow{\mathrm{Cay}}(D; y - x, z - x)$.

Lem. *$\exists$ ham cyc H in $\overrightarrow{\Delta}$, with $d_H(0, v + x)$ even (or $d_H(0, v + y)$ even or $d_H(0, v + z)$ even).*

Proof. m is even, and $v_1 + v_2 + v_3 \equiv -1 \pmod m$
$\Rightarrow$ some v_i is odd.

Wolog assume v_1 is odd.

Define

$D_0 = \{ (d_1, d_2, d_3) \in D \mid d_1 \text{ is even} \}$

$D_1 = \{ (d_1, d_2, d_3) \in D \mid d_1 \text{ is odd} \}$

Then $D_0 \cup D_1$ is a bipartition of $\overrightarrow{\Delta}$.

- $(0,0,0) \in D_0$
- $v + x = (\text{odd}, ?, ?) + x = (\text{even}, ?, ?) \in D_0$

$\Rightarrow d_H(0, v + x)$ is even for every cycle in $\overrightarrow{\Delta}$. $\square$

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