Introduction to Arithmetic Groups

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Abstract. This lecture is intended to introduce non-experts to this beautiful topic.

For further reading, see:

D. W. Morris, *Introduction to Arithmetic Groups*. Deductive Press, 2015.
http://arxiv.org/src/math/0106063/anc/

What is an arithmetic group?

Every *arithmetic group* Γ is a group of matrices with integer entries.

More precisely, $\Gamma \doteq \mathrm{SL}(n,\mathbb{Z}) \cap G =: G_{\mathbb{Z}}$ where

- $\mathrm{SL}(n,\mathbb{Z}) = \left\{ n \times n \text{ mats } (a_{ij}) \;\middle|\; \begin{matrix} a_{i,j} \in \mathbb{Z}, \\ \det = 1 \end{matrix} \right\}$
- $G \subseteq \mathrm{SL}(n,\mathbb{R})$ is connected Lie group semisimple, def'd over $\mathbb{Q}$

Examples

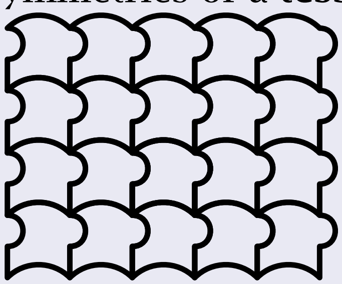
- $G = \mathrm{SL}(n,\mathbb{R}) \Rightarrow \Gamma = \mathrm{SL}(n,\mathbb{Z})$
- $G = \mathrm{SO}(1,n) = \mathrm{Isom}(x_1^2 - x_2^2 - \cdots - x_{n+1}^2)$
  $\Rightarrow \Gamma = \mathrm{SO}(1,n)_{\mathbb{Z}}$
- subgroup of Γ that has finite index

Geometric motivation

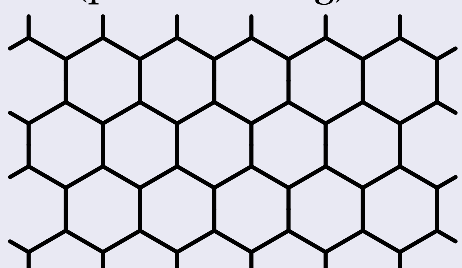
Group theory = the study of symmetry

Example

Symmetries of a **tessellation** (periodic tiling)



symmetry group $\Gamma = \mathbb{Z}^2$



$\Gamma \doteq \mathbb{Z}^2$

Thm (Bieberbach, 1910). $\forall$ tess of $\mathbb{R}^n$, $\Gamma \doteq \mathbb{Z}^n$.

Other spaces yield groups that are more interesting.

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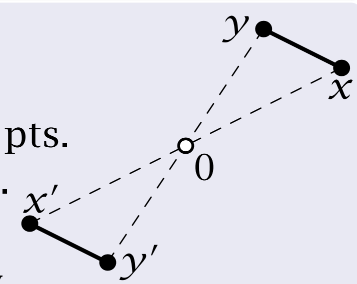
$\mathbb{R}^n$ is a **symmetric space**:



- **homogeneous**: every pt looks like all other pts.
  $\forall x, y,\ \exists$ isometry $x \mapsto y$.
- reflection through a point $(x' = -x)$ is an isometry.

Assume tiles of tess of X are compact (or finite vol). Then symmetry group Γ is a *lattice* in $\mathrm{Isom}(X) = G$:

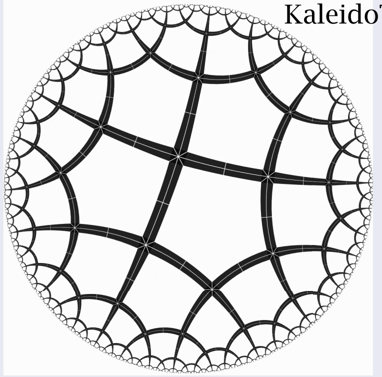
G/Γ is compact (or has finite volume).

Γ is **cocompact** or **uniform** Γ is **noncompact** or **nonuniform**

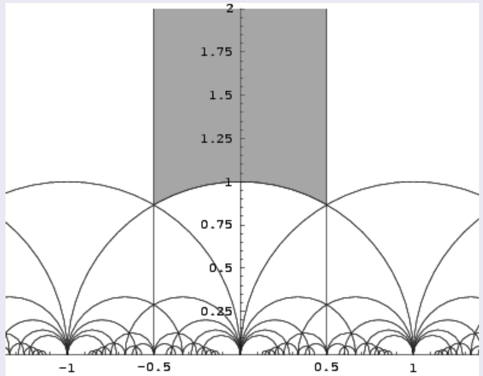
Many lattices in G are arithmetic subgroups.

Eg. Tess'ns of hyperbolic plane $\mathfrak{h}^2$. (symmetric space)

$G \doteq \mathrm{SO}(1,2) \doteq \mathrm{SL}(2,\mathbb{R})$.



KaleidoTile



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Γ = cocompact lattice $\qquad \Gamma = \mathrm{SL}(2,\mathbb{Z}) \doteq \mathrm{SO}(1,2)_{\mathbb{Z}}$

Tessellations of other symmetric spaces correspond to lattices in other interesting groups.

Other interesting groups G: simple Lie group.

- $\mathrm{SO}(m,n) = \mathrm{Isom}\left(\sum_{k=1}^m x_k^2 - \sum_{k=1}^n x_{m+k}^2\right)$
  (lattice: $G_{\mathbb{Z}}$)
- $\mathrm{SU}(m,n)$: change $\mathbb{R}$ to $\mathbb{C}$ and x_k^2 to $x_k\overline{x_k}$
  (lattice: $G_{\mathbb{Z}+\mathbb{Z}i}$)
- $\mathrm{Sp}(m,n)$: change $\mathbb{C}$ to $\mathbb{H}$
  (lattice: $G_{\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k} = G_{\mathbb{H}_{\mathbb{Z}}}$)
- $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$
- $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{H})$
- finitely many "exceptional grps" (E_6, E_7, E_8, F_4, G_2)

Semisimple Lie group $\doteq G_1 \times G_2 \times \cdots G_r$.

Arithmetic groups are lattices

Tessellations of other symmetric spaces correspond to lattices in semisimple Lie groups.

Theorem (Borel and Harish-Chandra)

Every arithmetic group is a lattice:
Semisimple Lie group G defined over $\mathbb{Q}$ $\Rightarrow$ $G_{\mathbb{Z}}$ is a lattice.

Remark

There are *many* arithmetic lattices in G:
Many ways to embed G in $\mathrm{SL}(n,\mathbb{R})$ (if n is large)
$\Rightarrow$ many different possibilities for $G_{\mathbb{Z}}$.
Some are cocpct, and some are not cocpct [Borel].

There are *many* arithmetic lattices in $\mathrm{SO}(m,n) = \mathrm{Isom}\left(\sum_{k=1}^m x_k^2 - \sum_{k=1}^n x_{m+k}^2\right)$.

Proof. Choose $a_1, a_2, \ldots, a_{m+n} \in \mathbb{Q}^+$.
Let $Q(x) = \sum_{k=1}^m a_k x_k^2 - \sum_{k=1}^n a_{m+k} x_{m+k}^2$.
$\mathbb{R}$ change of vars takes $Q(x)$ to the standard form.
Therefore $\mathrm{Isom}(Q(x)) \cong \mathrm{SO}(m,n)$,
so $\mathrm{Isom}(Q(x))_{\mathbb{Z}} \rightsquigarrow$ latt in $\mathrm{SO}(m,n)$. □

Fact. $\mathrm{Isom}(Q(x))_{\mathbb{Z}}$ *nonuniform*
$\Leftrightarrow$ $Q(x) = 0$ *has nonzero soln in* $\mathbb{Z}^{m+n}$.

Eg. $\mathrm{Isom}(x_1^2 - x_2^2 - x_3^2 - x_3^2)_{\mathbb{Z}}$ is noncocompact latt,
but $\mathrm{Isom}(7x_1^2 - x_2^2 - x_3^2 - x_3^2)_{\mathbb{Z}}$ is cocompact latt.

Theorem (Borel and Harish-Chandra)

Every arithmetic group is a lattice.

Converse:

Margulis Arithmeticity Theorem

Every lattice in simple G is an arithmetic subgroup unless $G \doteq \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$.

Remark (technicality)

Some cocpct latts use *algebraic integers* not in $\mathbb{Z}$.
Let $\alpha = \sqrt{2}$ (or other algebraic int) and $F = \mathbb{Q}[\alpha]$.
If G is defined over F and $G_{\mathbb{Z}[\alpha]}$ is discrete (no acc pts)
then $G_{\mathbb{Z}[\alpha]}$ is an arithmetic subgroup of G.
It is a lattice in G. ("Restriction of scalars")

Basic algebraic properties

Proposition

$G_{\mathbb{Z}}$ is finitely generated.

Proof when Γ is cocompact.

Choose bounded, open $C \subset G$, such that $G_{\mathbb{Z}}\, C = G$.
Let $S = \{\, s \in G_{\mathbb{Z}} \mid sC \cap C \neq \emptyset \,\}$.
S is finite, because $G_{\mathbb{Z}}$ is discrete.
$\langle S \rangle C$ is open and closed:
Acc pt $p \in \gamma C \quad \Rightarrow \quad \gamma C \cap \langle S \rangle C \neq \emptyset$
$\Rightarrow \quad C \cap \gamma^{-1} s_1 \cdots s_n C \neq \emptyset$
$\Rightarrow \quad \gamma^{-1} s_1 \cdots s_n \in S \quad \Rightarrow \quad \gamma \in \langle S \rangle$
$\Rightarrow \quad p \in \langle S \rangle C$.
So $\langle S \rangle C = G$. Preceding argument: $G_{\mathbb{Z}} = \langle S \rangle$. □

Lemma (Selberg)

*$G_{\mathbb{Z}}$ is **virtually torsion-free**:*
$\exists$ finite-index subgrp $\Gamma' \subseteq G_{\mathbb{Z}}$, Γ' is torsion-free.
(no nontrivial elements of finite order)

Proof.

Natural homo $\rho_3 \colon \Gamma \to \mathrm{SL}(n, \mathbb{Z}/3\mathbb{Z})$, $\Gamma' = \ker(\rho_3)$.
Since $\Gamma/\Gamma' \cong \rho_3(\Gamma) \subseteq \mathrm{SL}(n, \mathbb{Z}/3\mathbb{Z})$ is finite,
it suffices to show Γ' is torsion-free.

Let $\gamma \in \Gamma'$, write $\gamma = \mathrm{I} + 3^k T$, $T \not\equiv 0 \pmod 3$.

$$\begin{aligned} \gamma^m &= (\mathrm{I} + 3^k T)^m \\ &= \mathrm{I} + m(3^k T) + \binom{m}{2} 3^{2k} T^2 + \cdots \\ &\equiv \mathrm{I} + 3^k m T \pmod{3^{k+\ell+1}} \quad \text{if } 3^\ell \mid m \\ &\not\equiv \mathrm{I} \pmod{3^{k+\ell+1}} \qquad \text{if } 3^{\ell+1} \nmid m. \end{aligned}$$

□

Lemma. *The centre of Γ is finite.*

Exercise

Prove when Γ is a cocompact lattice in $G = \mathrm{SL}(n, \mathbb{R})$.

Hint. More precisely, show $Z(\Gamma) \subseteq Z(G)$.

For $z \in Z(\Gamma)$, the conjugacy class z^G is compact.

If $a = \mathrm{diag}(a_1, \ldots, a_n)$ is any diagonal(izable) element of G, then
$(a^{-n}\, z\, a^n)_{ij} = z_{ij} \cdot (a_j/a_i)^n$.
This is bounded, so it must be constant.
Therefore a centralizes z.

G is generated by its diagonalizable elements.