Real representations of $\mathfrak{sp}(n)$ have $\mathbb{Q}$-forms

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Abstract. Suppose $\{A_1, \ldots, A_\ell\}$ is a basis (over $\mathbb{R}$) of a Lie algebra $\mathfrak{g}$ of $m \times m$ matrices. If all the entries of each matrix $A_1, \ldots, A_\ell$ are rational, then it is easy to see that the corresponding structure constants of $\mathfrak{g}$ are rational; that is, $[A_i, A_j] = \sum_k \alpha_{ij}^k A_k$, with $\alpha_{ij}^k \in \mathbb{Q}$. The converse is not usually true, but it does hold (after changing to a different basis of $\mathbb{R}^m$) if $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(n)$, for some n.

For Lie algebras other than $\mathfrak{sp}(n)$, it would be interesting to understand which choices of the structure constants α_{ij}^k force $A_1, \ldots, A_\ell$ to be (similar to) rational matrices. This is relevant to the study of lattices in certain 2-step nilpotent Lie groups.

Eg. $f(x) = x^2 + 1$

- is irreducible over $\mathbb{R}$ (can't be factored)
- is reducible over $\mathbb{C}$ (can be factored)

$$f(x) = (x - i)(x + i)$$

Analogue in representation theory.

(All rep'ns are finite dimensional.)

Eg. $G = \mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right\}$

has obvious representation on $\mathbb{R}^2$.

- This is irreducible. (No line is inv't.)
- Complexification is reducible.

$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$

$\{(z, iz)\}$ is an invariant subspace:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} z \\ iz \end{bmatrix} = \begin{bmatrix} (\cos\theta)z + (\sin\theta)(iz) \\ (-\sin\theta)z + (\cos\theta)(iz) \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta + i\sin\theta)z \\ i(\cos\theta + i\sin\theta)z \end{bmatrix} = \begin{bmatrix} z' \\ iz' \end{bmatrix}$$

We consider G = simple group.

Prop. $G = \mathrm{SL}(n, \mathbb{R})$ *(or any $\mathbb{R}$-split group)*
$\Rightarrow$ *every irred $\mathbb{R}$-rep'n of G remains irred over $\mathbb{C}$.*

Proof. Show every $\mathbb{C}$-rep'n W of G has a $\mathbb{R}$-form.

- $A = \{$diagonal matrices$\}$
- λ = highest weight of W (w.r.t. A)

Every root of $\mathfrak{g}_{\mathbb{C}}$ is $\mathbb{R}$-valued on A (A is $\mathbb{R}$-split)

$\Rightarrow \lambda(A) \subset \mathbb{R}$

$\Rightarrow$ we can form **real** highest-weight module V

$\Rightarrow W = V \otimes_{\mathbb{R}} \mathbb{C}$. $\square$

At opposite extreme, assume G is **compact**.

E.g., $G = \mathrm{SO}(n)$.

Lem. *The following are equivalent:*

a) *every irred $\mathbb{R}$-rep'n of G remains irred over $\mathbb{C}$*

b) $\otimes_{\mathbb{R}}\mathbb{C}$: $\{\mathbb{R}$*-rep'ns of* $G\} \to \{\mathbb{C}$*-rep'ns of* $G\}$ *is an isomorphism of categories*

c) *every $\mathbb{C}$-rep'n of G has a $\mathbb{R}$-form (i.e., $\otimes_{\mathbb{R}}\mathbb{C}$ is onto)*

d) $\rho: G \to \mathrm{GL}(n, \mathbb{C}) \Rightarrow \rho(G) \subset \mathrm{GL}(n, \mathbb{R})$ *after a change of basis*

e) $\forall$ *irred $\mathbb{R}$-rep'n of G,* $\mathrm{End}_G(V) \cong \mathbb{R}$

Proof. ($a \Rightarrow c$): $W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is reducible:

$W_{\mathbb{R}} = V_1 \oplus V_2 \quad = V_1 \oplus iV_1$, so $W = V_1 \otimes_{\mathbb{R}} \mathbb{C}$.

($a \Leftrightarrow e$): Schur's Lemma.

V irred/$\mathbb{R} \Rightarrow \mathrm{End}_G(V)$ = div'n alg = $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$

$\Rightarrow \mathrm{End}_G(V_{\mathbb{C}}) = \mathrm{End}_G(V) \otimes_{\mathbb{R}} \mathbb{C}$

$\cong \mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$, or $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

Only $\mathbb{C}$ is a division alg: $V_{\mathbb{C}}$ irred $\Leftrightarrow \mathrm{End}_G(V) = \mathbb{R}$

($b \Leftrightarrow c$): $\otimes_{\mathbb{R}}\mathbb{C}$ is always one-to-one. $\square$

Assume G is **compact**. E.g., $G = \mathrm{SO}(n)$.

Lem. *W has $\mathbb{R}$-form $\Leftrightarrow \exists$ G-inv conj $\tau\colon W \to W$.*

Cor. *If W has a $\mathbb{R}$-form, then $\overline{W} \cong W$.*

Defn. $\overline{W}$ = same set of vectors as W,
but different scalar multiplication: $\alpha * v = \overline{\alpha} v$.

Rem. For $W = \mathbb{C}^n$, $\rho(G) \subset \mathrm{GL}(n,\mathbb{R})$,
$\overline{\rho}(g) = \overline{\rho(g)}$ (conjugate the matrix entries)

Proof of Cor. $\tau\colon W \to \overline{W}$ is isomorphism. $\square$

Lem. *$-$(highest weight of W) is a weight of $\overline{W}$.*

Proof. $\overline{\text{highest wt of } W}$ is a wt of $\overline{W}$.

$\mathfrak{t}$ = maximal torus of $\mathfrak{g}$
$\Rightarrow \lambda(\mathfrak{t}) \subset i\mathbb{R}$ (bcs $\mathfrak{g}$ is cpct)
$\Rightarrow \overline{\lambda}(t) = \overline{\lambda(t)} = -\lambda(t)$. $\square$

Lem. *W has $\mathbb{R}$-form $\Leftrightarrow \exists$ G-inv conj $\tau\colon W \to W$.*

- *τ is conjugate-linear*
  $\tau(\alpha_1 w_1 + \alpha_2 w_2) = \overline{\alpha_1}\,\tau(w_1) + \overline{\alpha_2}\,\tau(w_2)$.
- $\tau^2 = \mathrm{Id}$.
- $\tau(gw) = g\,\tau(w)$.

Proof. ($\Rightarrow$) Wolog $W = \mathbb{C}^n$, $\rho(G) \subset \mathrm{GL}(n,\mathbb{R})$.
$\overline{gw} = \overline{g}\,\overline{w} = g\,\overline{w}$ $\Rightarrow$ define $\tau(w) = \overline{w}$.

($\Leftarrow$) $(\tau^2 - 1) = 0$ $\Rightarrow$ eigenvalues $= \pm 1$.
$V_+ = +1$-eigenspace, $V_- = -1$-eigenspace
$W = V_+ \oplus V_- = V_+ \oplus iV_+$ $\Rightarrow W = V_+ \otimes_{\mathbb{R}} \mathbb{C}$. $\square$

Assume G is **compact**. E.g., $G = \mathrm{SO}(n)$.

Lem. *If W has a $\mathbb{R}$-form, then $\overline{W} \cong W$.*

Lem. *$-$(highest weight of W) is a weight of $\overline{W}$.*

Rem. W_λ has a $\mathbb{R}$-form
$\Rightarrow -\lambda$ is a wt of W_λ (lowest weight)
$\Rightarrow \mathbf{w}_0(\lambda) = -\lambda$ ($\mathbf{w}_0$ = long el't Weyl grp)

Cor. *If every $\mathbb{C}$-rep'n of G has a $\mathbb{R}$-form, then $\mathbf{w}_0(\lambda) = -\lambda$, $\forall\lambda$*
($\Leftrightarrow$ *each $g \in G$ is conj to its inverse g^{-1}*).

Eg. $\exists$ $\mathbb{C}$-rep'n of $\mathrm{SU}(n)$ with no $\mathbb{R}$-form ($n > 2$).

$$\begin{bmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \ddots \end{bmatrix} \not\sim \begin{bmatrix} 1/\omega_1 & & \\ & 1/\omega_2 & \\ & & \ddots \end{bmatrix}$$

Rem. $\mathbf{w}_0 = -\mathrm{Id}$ *except*: type A_n, D_{odd}, E_6.
($\mathbf{w}_0$ is the "opposition involution.")

Prop. *Every $\mathbb{C}$-rep'n of G has a $\mathbb{R}$-form if*
$\mathbf{w}_0(\lambda) = -\lambda$, $\forall\lambda$ and G is adjoint.

Proof. We have a representation of $\mathfrak{g}_{\mathbb{C}}$ on $\mathbb{C}^n$.

- Let $\mathfrak{g}_S \subset \mathfrak{g}_{\mathbb{C}}$ be a split $\mathbb{R}$-form,
- σ_G = conjugation on $\mathfrak{g}_{\mathbb{C}}$ corresp to $\mathfrak{g}$,
- σ_S = conjugation on $\mathfrak{g}_{\mathbb{C}}$ corresp to $\mathfrak{g}_S$.

Wolog $\rho(\mathfrak{g}_S) \subset \mathfrak{gl}(n,\mathbb{R})$, so $\overline{g\,w} = \sigma_S(g) \cdot \overline{w}$.
$\sigma_G \circ \sigma_S$ is $\mathbb{C}$-linear auto of $\mathfrak{g}_{\mathbb{C}}$
$\Rightarrow \sigma_G \circ \sigma_S = \mathrm{Ad}_{\mathfrak{g}} a$, $\exists a \in G_{\mathbb{C}}$ (outer aut??)

Define $\tau\colon \mathbb{C}^n \to \mathbb{C}^n$ by $\tau(w) = a \cdot \overline{w}$.

- $\tau(gw) = a\,\sigma_S(g)\,\overline{w} = \sigma_G(g)\,a\,\overline{w} = g\,\tau(w)$
- τ is conjugate-linear.
- $\tau^2(w) = a \cdot \overline{\tau(w)} = a \cdot \overline{a \cdot \overline{w}} = a \cdot \sigma_S(a) \cdot w = w$.
  $\mathrm{Id} = (\sigma_G)^2 = \mathrm{Ad}_{\mathfrak{g}}\big(a \cdot \sigma_S(a)\big)$
  $\Rightarrow a \cdot \sigma_S(a) \in Z(G) = \{e\}$. $\square$

Note. Proof shows W_λ has $\mathbb{R}$-form if
$\mathbf{w}_0(\lambda) = -\lambda$ and $\lambda \in \langle \text{roots} \rangle_{\mathbb{Z}}$ ($\mathbb{Z}$-span).

Prop (Tits). *Assume* $\mathbf{w}_0(\lambda) = -\lambda$.
Write $\lambda = \sum m_\alpha \cdot \alpha$, *linear comb of simple roots.*

- $\sum m_\alpha \in \frac{1}{2}\mathbb{Z}$ $\quad (2\lambda \in \langle \textit{roots} \rangle \Rightarrow m_\alpha \in \frac{1}{2}\mathbb{Z})$.
- W_λ *has a* $\mathbb{R}$*-form iff* $\sum m_\alpha \in \mathbb{Z}$.

Eg. Type B_n $\qquad$ $\mathrm{SO}(2n+1)$.

$$\lambda_\ell = \alpha_1 + 2\alpha_2 + \cdots + (\ell-1)\alpha_{\ell-1} + \ell\,(\alpha_\ell + \cdots + \alpha_n) \qquad (\ell < n)$$

$$\lambda_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)$$

λ_n has a $\mathbb{R}$-form $\quad \Leftrightarrow \quad n \equiv 0, 3 \pmod 4$
$\Leftrightarrow$ every $\mathbb{C}$-rep'n of $\mathrm{SO}(2n+1)$ has a $\mathbb{R}$-form.

Eg. Type C_n $\qquad$ cpct symplectic group $\mathrm{Sp}(2n)$.

$$\lambda_\ell = \alpha_1 + 2\alpha_2 + \cdots + (\ell-1)\alpha_{\ell-1} + \ell\alpha_\ell + \cdots + \ell\alpha_{n-1} + \frac{\ell}{2}\alpha_n$$

$\exists$ $\mathbb{R}$-form $\quad \Leftrightarrow \quad \ell$ even.
Not all $\mathbb{C}$-rep'ns have a $\mathbb{R}$-form.

G = compact, simple Lie group.

Tits: Every $\mathbb{C}$-rep'n of $\mathfrak{g}$ has a $\mathbb{R}$-form $\Leftrightarrow \langle \times\times\times\times \rangle$.

Eg. $f(x) = x^2 + 1$

- is irreducible over $\mathbb{R}$
- but reducible over $\mathbb{C}$

Eg. $f(x) = x^2 - 3$

- is irreducible over $\mathbb{Q}$
- but reducible over $\mathbb{R}$ $\qquad (x - \sqrt{3})(x + \sqrt{3})$

Study the field extension $\mathbb{R}/\mathbb{Q}$, *instead of* $\mathbb{C}/\mathbb{R}$.

Choose basis $E_1, \ldots, E_n$ of $\mathfrak{g}$,
such that the structure constants belong to $\mathbb{Q}$,
i.e., $[E_k, E_\ell] = \sum \alpha^j_{k,\ell} E_j$ $\quad$ with $\alpha^j_{k,\ell} \in \mathbb{Q}$.
Then $\mathbb{Q}$-span$(E_1, \ldots, E_n)$ is a Lie algebra over $\mathbb{Q}$;
it is a $\mathbb{Q}$*-form of* $\mathfrak{g}$.

$\exists$ $\mathbb{Q}$-form of $\mathfrak{g}$: $\qquad \langle x_\alpha - x_{-\alpha}, i(x_\alpha + x_{-\alpha}), ih^*_\alpha \rangle$.

Defn. $\mathbb{Q}$-Lie alg $\mathfrak{g}_\mathbb{Q}$ is a $\mathbb{Q}$*-form of* $\mathfrak{g}$ if $\mathfrak{g}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R} \cong \mathfrak{g}$.

Defn. $\mathfrak{g}_\mathbb{Q}$ is $\mathbb{R}$-universal:
every $\mathbb{R}$-rep'n of $\mathfrak{g}_\mathbb{Q}$ has a $\mathbb{Q}$-form.

- $\rho\colon \mathfrak{g}_\mathbb{Q} \to \mathfrak{gl}(n, \mathbb{R}) \qquad \Rightarrow \rho(\mathfrak{g}_\mathbb{Q}) \subset \mathfrak{gl}(n, \mathbb{Q})$
  after a change of basis in $\mathbb{R}^n$.

Ques. *Does* $\mathfrak{g}$ *have an* $\mathbb{R}$*-universal* $\mathbb{Q}$*-form?*

Ques. *Is* **every** $\mathbb{Q}$*-form* $\mathbb{R}$*-universal?*

Ques. *Which* $\mathbb{Q}$*-forms are* $\mathbb{R}$*-universal?*

Last one is still open — I plan to work on it.

Lem. *The following are equivalent:*

- *every irred* $\mathbb{Q}$*-rep of* $\mathfrak{g}_\mathbb{Q}$ *remains irred over* $\mathbb{R}$
- $\otimes_\mathbb{Q}\mathbb{R}\colon \{\mathbb{Q}\text{-}\textit{rep'ns of } \mathfrak{g}_\mathbb{Q}\} \to \{\mathbb{R}\text{-}\textit{rep'ns of } \mathfrak{g}\}$
  is an isomorphism of categories
- *every* $\mathbb{R}$*-rep'n of* $\mathfrak{g}_\mathbb{Q}$ *has a* $\mathbb{Q}$*-form*
  (i.e., $\otimes_\mathbb{Q}\mathbb{R}$ *is onto)*
- $\rho\colon \mathfrak{g}_\mathbb{Q} \to \mathfrak{gl}(n, \mathbb{R}) \Rightarrow \rho(\mathfrak{g}_\mathbb{Q}) \subset \mathfrak{gl}(n, \mathbb{Q})$
  after a change of basis
- $\forall$ *irred* $\mathbb{Q}$*-rep'n of* $\mathfrak{g}_\mathbb{Q}$, $\mathrm{End}_{\mathfrak{g}_\mathbb{Q}}(V) \otimes_\mathbb{Q} \mathbb{R}$
  is a division algebra.

Rem. W_λ has a $\mathbb{Q}$-form if $\lambda \in \langle \text{roots} \rangle_\mathbb{Z}$
and $\mathfrak{g}_\mathbb{Q}$ splits over a quadratic ext'n of $\mathbb{Q}$.
(Same proof as for $\mathbb{C}/\mathbb{R}$.)

Geometric motivation.

Thm (Malcev). *N = simply conn nilp Lie grp. N has a lattice subgrp (Γ discrete, N/Γ cpct) $\Leftrightarrow$ $\mathfrak{n}$ has a $\mathbb{Q}$-form.*

Easiest case: N is 2-step nilpotent ($N/Z(N)$ abel).

Construction. Given
- $\mathbb{R}$-subspace $Z \subset \mathfrak{so}(n)$,
- inner product on $\mathfrak{so}(n)$:
  $$\langle z_1 \mid z_2 \rangle = -\mathrm{trace}(z_1 z_2).$$

Let $\mathfrak{n} = \mathbb{R}^n \oplus Z$, with $Z \subset \mathfrak{z}(\mathfrak{n})$:
- $[x,y] \in Z$, defined by $\langle [x,y] \mid z \rangle = z(x) \cdot y$.

$\mathfrak{n}$ has a $\mathbb{Q}$-form
$\Leftrightarrow$ $\exists$ bases $x_1, \cdots, x_n$ of $\mathbb{R}^n$, $z_1, \ldots, z_p$ of Z,
$z_i(x_j) \cdot x_k \in \mathbb{Q}$.

I.e., $z_i \in \mathfrak{gl}(n, \mathbb{Q})$ — exactly our question.

¿More on this tomorrow in Pat Eberlein's talk?

Ques. Does $\mathfrak{g}$ have an $\mathbb{R}$-universal $\mathbb{Q}$-form?

Thm (Raghunathan, Eberlein, Pink-Prasad). *Yes, the "standard" $\mathbb{Q}$-form is $\mathbb{R}$-universal:*
$$\mathfrak{g}_{\mathbb{Q}} = \langle x_\alpha - x_{-\alpha}, i(x_\alpha + x_{-\alpha}), i h_\alpha^* \rangle.$$

Proof (Pink-Prasad).
- V = irred $\mathbb{Q}$-rep'n of $\mathfrak{g}_{\mathbb{Q}}$.
- $D = \mathrm{End}_{\mathfrak{g}_{\mathbb{Q}}}(V)$ = division algebra

$\mathfrak{g}_{\mathbb{Q}}$ splits over $\mathbb{Q}(i)$ (bcs $\mathfrak{g}_{\mathbb{Q}(i)} \supset$ Chev. basis)
$\Rightarrow D$ splits over $\mathbb{Q}(i)$
(i.e., $D \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \mathrm{Mat}_{2\times 2}\big(\mathbb{Q}(i)\big)$
and D is a quaternion algebra (deg. 2)

$\mathfrak{g}_{\mathbb{Q}}$ is (quasi-)split over $\mathbb{Q}_p$ (for all $p \neq 2$)
$\Rightarrow D$ splits over $\mathbb{Q}_p$

V is reducible over $\mathbb{R}$ $\Rightarrow D$ splits over $\mathbb{R}$
$\Rightarrow D$ splits at *all* places $\Rightarrow D$ splits over $\mathbb{Q}$.
$\rightarrow\leftarrow$ (D = division algebra) $\square$

Ques. *Is* **every** *$\mathbb{Q}$-form $\mathbb{R}$-universal?*

Prop. *Every $\mathbb{Q}$-form of $\mathfrak{g}$ is $\mathbb{R}$-universal if*
$\mathfrak{g} = \mathfrak{sp}(n)$ or $\mathfrak{so}(8n \pm 3)$
or any exceptional, except perhaps E_6.

Note. Although not obvious, every $\mathbb{Q}$-form of $\mathfrak{sp}(n)$ or of $\mathfrak{so}(2n+1)$ splits over some quadratic extension of $\mathbb{Q}$.

Complement. There exists a $\mathbb{Q}$-form of $\mathfrak{g}$ that
- is **not** $\mathbb{R}$-universal, and
- splits over some quadratic extension of $\mathbb{Q}$

$\Leftrightarrow$ either
- $\mathfrak{g} \cong \mathfrak{su}(2n)$, with $2n \geq 4$, or
- $\mathfrak{g} \cong \mathfrak{so}(n)$, with $n \not\equiv \pm 3 \pmod 8$.

Proof that every $\mathbb{Q}$-form of $\mathfrak{sp}(n)$ is $\mathbb{R}$-universal.

Spse $\exists$ irred $\mathbb{Q}$-rep'n V of $\mathfrak{g}_{\mathbb{Q}}$, s.t. $V \otimes_{\mathbb{Q}} \mathbb{R}$ is red:
$V_{\mathbb{R}} = X \oplus Y$.

Then $X \otimes_{\mathbb{R}} \mathbb{C}$ and $Y \otimes_{\mathbb{R}} \mathbb{C}$ are irred
(bcs $\mathfrak{g}_{\mathbb{Q}}$ splits over quadratic extension).

Let λ = highest weight of $X_{\mathbb{C}}$
$= \sum m_j \cdot \alpha_j$ (lin comb of simple roots).

Fundamental weights of $\mathfrak{sp}(n)$ (type C_n):
$$\lambda_\ell = \alpha_1 + 2\alpha_2 + \cdots + (\ell-1)\alpha_{\ell-1} + \ell\alpha_\ell + \ell\alpha_{\ell+1} + \cdots + \ell\alpha_{n-1} + \frac{\ell}{2}\alpha_n.$$
$\Rightarrow m_1, m_2, \ldots, m_{n-1} \in \mathbb{Z}$.

$X_{\mathbb{C}}$ has $\mathbb{R}$-form (i.e., X) $\Rightarrow \sum m_j \in \mathbb{Z}$ [Tits]
$\Rightarrow m_n \in \mathbb{Z}$
$\Rightarrow \lambda \in \mathbb{Z}$-span(roots)
$\Rightarrow X_{\mathbb{C}}$ has a $\mathbb{Q}$-form (as for $\mathbb{C}/\mathbb{R}$).

$V \otimes_{\mathbb{Q}} \mathbb{R} = X \oplus Y \cong (X_{\mathbb{Q}} \oplus Y_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R}$
$\Rightarrow V \cong X_{\mathbb{Q}} \oplus Y_{\mathbb{Q}}$ is reducible. $\rightarrow\leftarrow$ $\square$

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