$\mathbb{Q}$-forms of real representations of compact semisimple Lie groups

(after Raghunathan and Eberlein)

Dave Witte

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078

Notation.
- $\mathfrak{g}$ = real semisimple Lie algebra
  (can also state results for G)
- W = (finite dim'l) $\mathbb{C}$-representation of $\mathfrak{g}$
- V = (finite dim'l) $\mathbb{R}$-representation of $\mathfrak{g}$

Defn. $\mathbb{R}$-form of W = $\mathbb{R}$-subspace V of W, s.t.
- V is $\mathfrak{g}$-invariant;
- V = $\mathbb{R}$-span of $\mathbb{C}$-basis of W
  $\Leftrightarrow W = V \oplus iV$
  $\Leftrightarrow W \cong V \otimes_{\mathbb{R}} \mathbb{C} \qquad =: V_{\mathbb{C}}$

Prop. *$\mathfrak{g}$ $\mathbb{R}$-split*
(i.e., $\mathfrak{g}$ = $\mathbb{R}$-span of Chev. basis of $\mathfrak{g}_{\mathbb{C}}$)
$\Rightarrow$ every $\mathbb{C}$-repn of $\mathfrak{g}$ has a $\mathbb{R}$-form.

Proof. Wma W irred (bcs comp'ly red'ble).
λ = highest weight of W.
$v \in W_\lambda$.
$\lambda(\mathfrak{h}) \subset \mathbb{R} \quad \Rightarrow U(\mathfrak{g})v$ is proper $\mathbb{R}$-submodule. □

Notation. Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$:
$\{h_\alpha\}_{\alpha\in\Phi^+} \cup \{x_\alpha\}_{\alpha\in\Phi}$:
- $h_\alpha \in \mathfrak{h}_{\mathbb{C}}$;
- $h_\alpha^* = 2h_\alpha/\kappa(h_\alpha, h_\alpha)$;
- $x_\alpha \in (\mathfrak{g}_{\mathbb{C}})_\alpha$;
- $\alpha(t) = \kappa(t, h_\alpha)$, $\forall t \in \mathfrak{h}_{\mathbb{C}}$;
- $[x_\alpha, x_\beta] = \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ -h_\alpha^* & \text{if } \alpha+\beta = 0, \\ 0 & \text{if } 0 \neq \alpha+\beta \notin \Phi \text{ .} \end{cases}$

Cor. *$\mathfrak{g}_{\mathbb{C}}$ has a $\mathbb{R}$-form $\mathfrak{g}_0$, s.t.*
every $\mathbb{C}$-repn of $\mathfrak{g}_{\mathbb{C}}$ has a $\mathbb{R}$-form.

Rem. Today: replace $\mathbb{C}$ and $\mathbb{R}$ with $\mathbb{R}$ and $\mathbb{Q}$
(especially when $\mathfrak{g}$ is compact).

Lem (Schur). *V irred $\Rightarrow$*
- *$\operatorname{End}_{\mathfrak{g}}(V) \cong \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*
- *$\operatorname{End}_{\mathfrak{g}}(V) \cong \mathbb{R} \Leftrightarrow V_{\mathbb{C}}$ is irred.*

Proof. V irred $\Leftrightarrow \operatorname{End}_{\mathfrak{g}}(V)$ is division algebra.

The only div algs over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

$\operatorname{End}_{\mathfrak{g}}(V) = \operatorname{End}_{\mathfrak{g}}(V) \otimes_{\mathbb{R}} \mathbb{C}$
- $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}$ — div alg
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ — not div alg
- $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$ — not div alg □

Lem. *(a) Every $\mathbb{C}$-repn W of $\mathfrak{g}$ has a $\mathbb{R}$-form*
$\Leftrightarrow$ *(b) $V_{\mathbb{C}}$ is irred, $\forall$ irred $\mathbb{R}$-repn V*
$\Leftrightarrow$ *(c) $W|_{\mathbb{R}}$ is reducible, $\forall$ $\mathbb{C}$-repn W*

Proof. $(a \Rightarrow b)$ $V_{\mathbb{C}}$ reducible
$\Rightarrow V \otimes_{\mathbb{R}} \mathbb{C} \cong W_1 \oplus W_2$
$\cong (V_1 \otimes_{\mathbb{R}} \mathbb{C}) \oplus (V_1 \otimes_{\mathbb{R}} \mathbb{C})$
$\cong (V_1 \oplus V_2) \otimes_{\mathbb{R}} \mathbb{C}$
$\Rightarrow V \cong V_1 \oplus V_2$ is reducible. $\rightarrow\leftarrow$

$(a \Rightarrow c)$ $\mathbb{R}$-form of W is a $\mathbb{R}$-submodule.

$(b \Rightarrow a)$ Wma W irred.
$W \otimes_{\mathbb{R}} \mathbb{C}$ reducible
$\Rightarrow W|_{\mathbb{R}}$ reducible $\quad (c)$
$\Rightarrow W|_{\mathbb{R}} \cong V_1 \oplus V_2$
$\Rightarrow W = V_1 \oplus iV_1$
so V_1 is a $\mathbb{R}$-form of W. $\square$

Eg. The $2D$ repn of SU(2) has **no** $\mathbb{R}$-form.

Analogously:
- $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-form of $\mathfrak{g}$
  $= \mathbb{Q}$-subalgebra, $\mathbb{Q}$-span of $\mathbb{R}$-basis of $\mathfrak{g}$
- $\mathbb{Q}$-form of $V = \mathfrak{g}_{\mathbb{Q}}$-inv, $\mathbb{Q}$-span of $\mathbb{R}$-basis

Lem. *Every $\mathbb{R}$-repn V of $\mathfrak{g}$ has a $\mathbb{Q}$-form*
$\Leftrightarrow$ *$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is irred, $\forall$ irred $\mathbb{Q}$-repn $V_{\mathbb{Q}}$ of $\mathfrak{g}_{\mathbb{Q}}$*
$\Leftrightarrow$ $\mathrm{End}_{\mathfrak{g}_{\mathbb{Q}}}(V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}, \mathbb{C}, \mathbb{H}$, *$\forall$ irred $V_{\mathbb{Q}}$*

Prop. $\mathfrak{g}_{\mathbb{Q}}$ *$\mathbb{Q}$-split (i.e., $\mathbb{Q}$-span of Chev. basis)*
$\Rightarrow$ *every $\mathbb{R}$-repn of $\mathfrak{g}$ has a $\mathbb{Q}$-form.*

Cor. $\mathfrak{g}$ *split* $\Rightarrow$
$\exists$ $\mathfrak{g}_{\mathbb{Q}}$, s.t. every $\mathbb{R}$-repn of $\mathfrak{g}$ has a $\mathbb{Q}$-form.

Thm (Raghunathan, Eberlein). $\mathfrak{g}$ *compact* $\Rightarrow$
$\exists$ $\mathfrak{g}_{\mathbb{Q}}$, s.t. every $\mathbb{R}$-repn of $\mathfrak{g}$ has a $\mathbb{Q}$-form.

$\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-span of $\{ih_\alpha, x_\alpha + x_{-\alpha}, i(x_\alpha - x_{-\alpha})\}$.

Cor. $\forall \mathfrak{g}$, $\exists$ $\mathfrak{g}_{\mathbb{Q}}$, *s.t. every $\mathbb{R}$-repn of $\mathfrak{g}$ has $\mathbb{Q}$-form.*

Thm (Raghunathan). *Assume*
- $\mathfrak{g}$ *compact;*
- $\mathfrak{g}_{\mathbb{Q}}$ *is $\mathbb{Q}[i]$-split;*
- *longest el't of Weyl grp of $\mathfrak{g}$ is def'd over $\mathbb{Q}$.*

Then every $\mathbb{R}$-repn of $\mathfrak{g}$ has a $\mathbb{Q}$-form.

Rem. The obvious compact $\mathbb{Q}$-form is $\mathbb{Q}[i]$-split, and every el't of the Weyl group is def'd over $\mathbb{Q}$.

Eg. $\exists$ $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{Q}}$, W, s.t.
- W has a $\mathbb{R}$-form;
- W has a $\mathbb{Q}[i]$-form;
- W does **not** have a $\mathbb{Q}$-form.

Can take $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ (split) or $\mathfrak{so}(8)$ (compact).

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$: in alg'ic terms, $\exists$ quat. alg D over $\mathbb{Q}$, s.t. D is div alg, but splits over $\mathbb{R}$ and $\mathbb{Q}(i)$.

Concrete construction: Let $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-span of
$$\left\{ \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix} \right\}$$
$\mathbb{R}$-basis of $\mathfrak{g}$, brackets in $\mathbb{Q} \Rightarrow \mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form.

2D real repn of $\mathfrak{g}$ defined over $\mathbb{Q}$
$\Rightarrow \exists\, g \in \mathrm{GL}(2,\mathbb{R})$, s.t. $g^{-1}\mathfrak{g}_{\mathbb{Q}} g \subset \mathfrak{sl}(2,\mathbb{Q})$.
$\Rightarrow g^{-1}\mathfrak{g}_{\mathbb{Q}} g = \mathfrak{sl}(2,\mathbb{Q})$.
El'ts of $\mathfrak{g}_{\mathbb{Q}}$ have nonzero det (bcs $3 \neq a^2 + b^2$), but $\mathfrak{sl}(2,\mathbb{Q})$ has elements of determinant 0. $\rightarrow\leftarrow$

Mat ent's of $\mathfrak{g}_{\mathbb{Q}}$ in $\mathbb{Q}(i)$ for basis $\{(1,i), \sqrt{3}(1,-i)\}$.

Lem. *Assume*

- $\mathfrak{g}$ *compact;*
- $V \otimes_{\mathbb{R}} \mathbb{C} \cong W \oplus W$ *(irred);*
- $\lambda \colon \mathfrak{t} \to \mathbb{C}$ *highest weight of* $V_{\mathbb{C}}$*; and*
- $w \in N_G(\mathfrak{t})$ *longest el't of the Weyl group.*

Then

$$w^2|_{V_{\mathbb{C}}^{\lambda}} = \begin{cases} \mathrm{Id} & \text{if } V \text{ is reducible;} \\ -\mathrm{Id} & \text{if } V \text{ is irreducible .} \end{cases}$$

Proof only uses the fact that $\mathfrak{g}$ splits over $\mathbb{R}[i]$:

Lem. *Assume*

- $\mathfrak{g}_{\mathbb{Q}}$ *splits over* $\mathbb{Q}[i]$*, and* $\mathfrak{g}$ *is compact;*
- $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}[i] \cong U \oplus U$ *(irred);*
- $\mathfrak{t}_{\mathbb{Q}}$ *= max'l torus of* $\mathfrak{g}_{\mathbb{Q}}$ *(split over* $\mathbb{Q}[i]$*);*
- $\lambda \colon \mathfrak{t}_{\mathbb{Q}} \to \mathbb{Q}[i]$ *highest weight; and*
- $w \in N_G(\mathfrak{t})_{\mathbb{Q}}$ *longest el't of the Weyl group.*

Then

$$w^2|_{V_{\mathbb{C}}^{\lambda}} = \begin{cases} \mathrm{Id} & \text{if } V_{\mathbb{Q}} \text{ is reducible;} \\ -\mathrm{Id} & \text{if } V_{\mathbb{Q}} \text{ is irreducible .} \end{cases}$$

Thm (Raghunathan). *Assume*

- $\mathfrak{g}$ *compact;*
- $\mathfrak{g}_{\mathbb{Q}}$ *is* $\mathbb{Q}[i]$*-split;*
- *longest el't of Weyl grp of* $\mathfrak{g}$ *is def'd over* $\mathbb{Q}$.

Then every $\mathbb{R}$*-repn of* $\mathfrak{g}$ *has a* $\mathbb{Q}$*-form.*

Proof. Let $V_{\mathbb{Q}}$ = irred $\mathbb{Q}$-repn of $\mathfrak{g}_{\mathbb{Q}}$.
(We wish to show $V_{\mathbb{R}}$ is irred.)
$V_{\mathbb{C}}$ is either irred or sum of two irreds
(bcs $\mathfrak{g}_{\mathbb{Q}}$ splits over $\mathbb{Q}[i]$).

Case 1. $V_{\mathbb{C}}$ *is irred.* Then $V_{\mathbb{R}}$ is irred.

Case 2. $V_{\mathbb{C}}$ *= sum of two isomorphic irreds.*
We use the two lemmas:
$V_{\mathbb{Q}}$ irred
$\Rightarrow w^2|_{V_{\mathbb{C}}^{\lambda}} = -\mathrm{Id}$
$\Rightarrow V_{\mathbb{R}}$ is irreducible.

Case 3. $V_{\mathbb{C}}$ *= sum of two different irreds.*
Let $\mathcal{C} = \mathrm{End}_{\mathfrak{g}}(V)$.
This case: $\mathrm{End}_{\mathfrak{g}_{\mathbb{Q}}}(V_{\mathbb{Q}[i]}) \cong \mathbb{Q}[i] \oplus \mathbb{Q}[i]$.
Thus, $\mathcal{C}$ = div alg, s.t. $\mathcal{C} \otimes \mathbb{Q}[i] \cong \mathbb{Q}[i] \oplus \mathbb{Q}[i]$.

- $\mathbb{Q}[i] \oplus \mathbb{Q}[i]$ commutative $\Rightarrow \mathcal{C}$ is a field.
- $\dim_{\mathbb{Q}[i]}(\mathbb{Q}[i] \oplus \mathbb{Q}[i]) = 2 \Rightarrow \dim_{\mathbb{Q}} \mathcal{C} = 2$.
- $\mathcal{C} \otimes \mathbb{Q}[i] = \mathcal{C}[\sqrt{-1}]$ not field, $\Rightarrow i \in \mathcal{C}$.

Therefore $\mathcal{C} \cong \mathbb{Q}[i]$.

So

$$\mathrm{End}_{\mathfrak{g}}(V_{\mathbb{R}}) \cong \mathcal{C} \otimes \mathbb{R} \cong \mathbb{Q}[i] \otimes \mathbb{R} \cong \mathbb{C}$$

is a field.
Schur's Lemma: $V_{\mathbb{R}}$ is irreducible. $\square$

Proof of Raghunathan's Lemma ($w^2|_{V_{\mathbb{C}}^{\lambda}} = \pm\mathrm{Id}$).

Step 1. $\mathcal{C} = \mathrm{End}_{\mathfrak{g}}(V)$ *4-dim'l over* $\mathbb{R}$*, and*
V *is irred* $\Leftrightarrow$ $\mathcal{C}$ *is a div alg.*
$\mathrm{End}_{\mathfrak{g}}(V_{\mathbb{C}}) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$ is 4D + Schur's Lemma.

Step 2. $V_{\mathbb{C}}^{\lambda} + V_{\mathbb{C}}^{-\lambda}$ *is a 4D subspace, def'd over* $\mathbb{R}$.
Eigenvals of $\mathfrak{t}$ purely imag $\Rightarrow \overline{\lambda(t)} = -\lambda(t)$.
So $V_{\mathbb{C}}^{\lambda} + V_{\mathbb{C}}^{-\lambda}$ is defined over $\mathbb{R}$.

$V_{\mathbb{C}}$ = sum of two isomorphic irreds $\Rightarrow$ 4D.

Step 3. $E = (V_{\mathbb{C}}^{\lambda} + V_{\mathbb{C}}^{-\lambda}) \cap V$ *is faithful* $\mathcal{C}$*-module.*
$\mathcal{C}$ centralizes $\mathfrak{g}$ $\Rightarrow$ weight space is $\mathcal{C}$-invariant.
$V_{\mathbb{C}}^{\lambda}$ generates $V_{\mathbb{C}}$ (as $\mathfrak{g}$-module) $\Rightarrow E$ is faithful.

Step 4. $w^2|_{V_{\mathbb{C}}^{\lambda}}$ *is either* Id *or* $-$Id.
$w^2(\Phi^+) = w(\Phi^-) = \Phi^+ \Rightarrow w^2 \in T$.
$w(\lambda) = -\lambda \Rightarrow w(\lambda)(t) = 1/\big(\lambda(t)\big)$ for $t \in T$.

$$\lambda(w^2) = \frac{1}{w(\lambda)(w^2)} = \frac{1}{\lambda(w^{-1}w^2w)} = \frac{1}{\lambda(w^2)}.$$

Thus, $\lambda(w^2) = \pm 1$.

Step 5. E is invariant under w and $\mathfrak{t}$, so

- *$\{w\} \cup \mathfrak{t}$ gens a subalg $\mathcal{A}$ of $\operatorname{End}_{\mathbb{R}}(E)$, and*
- $\mathcal{A} \cong \begin{cases} \mathrm{Mat}_{2\times 2}(\mathbb{R}) & \textit{iff } w^2|_{V_{\mathbb{C}}^{\lambda}} = \mathrm{Id} \ , \\ \mathbb{H} & \textit{iff } w^2|_{V_{\mathbb{C}}^{\lambda}} = -\mathrm{Id} \ . \end{cases}$

Invariance is clear.

Image of $\mathfrak{t}$ in $\mathcal{A}$ is $i\mathbb{R} \subset \mathbb{C}$, negated by w.

$\mathbb{C}[w] \cong \mathbb{H}$ or $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

Step 6. $\operatorname{End}_{\mathcal{A}}(E)$ *is (anti)isomorphic to* $\mathcal{A}$.

Define $L, R \colon \mathcal{A} \to \operatorname{End}_{\mathbb{R}}(\mathcal{A})$ by

$L(a)x = ax$ and $R(a)x = xa$.

Then $\operatorname{End}_{L(\mathcal{A})}(\mathcal{A}) = R(\mathcal{A})$,

so suffices to show $E \cong L$ (as $\mathcal{A}$-modules).

$\mathcal{A} = \mathbb{H}$: $\dim_{\mathbb{R}} E = 4$

$\Rightarrow E =$ 1D vector space over $\mathcal{A}$

$\Rightarrow E \cong L$.

$\mathcal{A} \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$: unique nontriv (2D) irred

$\Rightarrow E \cong L$ (same dim, no trivial submods).

Step 7. The algebra $\mathcal{C}$ is (anti)isomorphic to $\mathcal{A}$.

E faithful $\Rightarrow \mathcal{C} \cong \mathcal{C}|_E$.

Defn of $\mathcal{C} \Rightarrow \mathcal{C}|_E \subset \operatorname{End}_{\mathcal{A}}(E) \cong^* \mathcal{A}$.

$\dim_{\mathbb{R}} \mathcal{C}|_E = 4 = \dim_{\mathbb{R}} \mathcal{A} \Rightarrow \mathcal{C}|_E \cong \dim_{\mathbb{R}} \mathcal{A}$.

Step 8. Completion of proof.

Combine Steps 1, 7, and 5:

V is irred $\Leftrightarrow \mathcal{C}$ is a div alg

$\Leftrightarrow \mathcal{A}$ is a div alg

$\Leftrightarrow w^2|_{V_{\mathbb{C}}^{\lambda}} = -\mathrm{Id}$. $\square$

References

The theorem in Section 3 of

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