$\mathrm{SL}(n,\mathbb{Q})$ has no volume-preserving actions on $(n-1)$-dimensional compact manifolds

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Abstract

As part of the "Zimmer program," numerous authors have studied volume-preserving actions of the group $\mathrm{SL}(n,A)$ on compact manifolds, where A is either the ring $\mathbb{Z}$ of integers or the field $\mathbb{R}$ of real numbers. On the other hand, very little seems to be known about the intermediate case where A is the field $\mathbb{Q}$ of rational numbers. As a first step in this direction, we show that $\mathrm{SL}(n,\mathbb{Q})$ has no nontrivial, C^∞, volume-preserving action on any compact manifold of dimension strictly less than n. The proof has two main ingredients: a theorem of Zimmer tells us that the action of any "S-arithmetic" subgroup must extend (a.e.) to a measurable action of its profinite completion, and the Congruence Subgroup Property provides a very nice description of this profinite completion. This is joint work with Robert J. Zimmer of the University of Chicago.

Transformation groups

Given: group G, (compact, C^∞) manifold M.

¿ What are the actions of G on M ?

I.e.: ¿ What are homos $\phi\colon G \to \mathrm{Diff}(M)$?

Question

¿ $\exists$ (*almost* faithful) *action* ?

Question

¿ $\exists$ *action of G on some k-dimensional M ?*

G acts on $M^k \implies G$ acts on $M^k \times S^1 = M^{k+1}$

Zimmer program

G is *large*.

E.g., G = (noncompact) simple Lie group $= \mathrm{SL}(n,\mathbb{R})$.

¿ $\exists$ action of $G = \mathrm{SL}(n,\mathbb{R})$ on some M^k ?

Answer

$\mathrm{SL}(n,\mathbb{R})$ acts on $\mathbb{R}P^{n-1}$, but not on any M^{n-2}.

Harder: actions must be *volume-preserving*.

M^k has G-invariant k-form (nowhere-vanishing)

Answer

$\mathrm{SL}(n,\mathbb{R})$ acts on $\mathrm{SL}(n,\mathbb{R})/\Gamma = M^{n^2-1}$, not on M^{n^2-2}.

Prop. $\mathrm{SL}(n,\mathbb{R})$ acts on M (vol-pres) $\Rightarrow \dim M \geq \dim G$.

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Poincaré Recurrence Theorem

$U, gU, g^2U, \ldots$ disjoint

$\Rightarrow \mathrm{vol}(\bigcup g^iU) = \sum \mathrm{vol}(g^iU) = \sum \mathrm{vol}(U) = \infty$.

$\Rightarrow \mathrm{vol}(M) = \infty$. $\quad \rightarrow\leftarrow$

$\therefore$ a.e. $x \in M$ is *recurrent* $\quad (\exists i,\ g^ix \approx x)$

Choose $g \in G$, $x \in \mathbb{R}^n \setminus \{0\}$ with $gx = \lambda x$ and $\lambda > 1$.

Then $g^ix = \lambda^i x \to \infty \not\approx x$.

G/H has finite volume $\Rightarrow G/N_G(H^\circ)$ has finite vol.

But $G/N_G(H^\circ)$ is **algebraic variety** $\hookrightarrow \mathbb{R}^m$ (or $\mathbb{R}P^m$).

$\therefore N_G(H^\circ) = G \quad \Rightarrow H^\circ = e \quad \Rightarrow \dim(G/H) = \dim G$.

¿ $\exists$ action of $G = \mathrm{SL}(n,\mathbb{R})$ on some M^k ? (vol-pres)

Answer

$\mathrm{SL}(n,\mathbb{R})$ acts on $\mathrm{SL}(n,\mathbb{R})/\Gamma = M^{n^2-1}$, not on M^{n^2-2}.

Very hard

Replace $\mathrm{SL}(n,\mathbb{R})$ (connected) with $\mathrm{SL}(n,\mathbb{Z})$ (discrete).

Example: $\mathrm{SL}(n,\mathbb{Z})$ acts on $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$.

Conjecture (Zimmer, 1984)

- $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ *or* $\mathrm{SL}(n,\mathbb{Z}[1/m])$, $n \geq 3$
- *Γ acts on compact mfld M^k, preserving volume*
- $k < n$

$\implies \Gamma$ *acts trivially.* (*Γ-action factors through finite group*)

Conjecture (Zimmer, 1984)

$\mathrm{SL}(n,\mathbb{Z})$ *does not act on M^{n-1}, preserving volume.*

Remark

- Easy for $k = 0, 1$.
- Known for $k = 2$ [Polterovich, Franks-Handel, Farb-Shalen]

Conjecture (Zimmer (?))

For $k = 2$, Γ can be any Kazhdan group.

Theorem (Margulis, 1974)

$\mathrm{SL}(n,\mathbb{Z})$ *does not act on M^k, preserving metric* (*if $n \geq 3$*).

I.e., $\varphi\colon \Gamma \to$ cpct Lie grp $\mathrm{SO}(N) \implies \varphi(\Gamma)$ finite.

• special case of Margulis Superrigidity Theorem

Thm. $\varphi\colon \mathrm{SL}(n,\mathbb{Z}) \to \mathrm{SO}(N) \implies \varphi(\Gamma)$ finite (if $n \geq 3$).

Exer. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \lambda^n v_+$, $\quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \lambda^n v_-$.

$\therefore$ For $v \in \mathbb{Z}^2$, $nv = \sum \pm g^{n_i} e_1$ with $\sum |n_i| < C \log n$.

For $\hat{g} = \begin{bmatrix} g & \\ & 1 \end{bmatrix}$ and $\overline{v} = \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix}$, $\quad \hat{g}^n \, \overline{v} \, \hat{g}^{-n} = \overline{g^n v}$.

Cor. Word length of $\overline{v}^n = \overline{nv}$ grows logarithmically.

Exer. $g \in \varphi(\Gamma)$ (& $|g| = \infty$) $\Rightarrow$ word len of g^n is linear.

Hint: λ = eigenval of g

$\Rightarrow \exists\, \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $|\sigma(\lambda)| > 1$.

Eigenval of $\sigma(g)^n = \sigma(\lambda)^n$ grows exponentially.

Theorem (Margulis, 1974)

$\mathrm{SL}(n,\mathbb{Z})$ *does not act on M^k, preserving metric* (*if $n \geq 3$*).

I.e., $\varphi\colon \Gamma \to$ cpct Lie grp $\mathrm{SO}(N) \implies \varphi(\Gamma)$ finite.

Warning. *Some* cocompact lattices *do* have homos to $\mathrm{SO}(N)$.

Conjecture (Zimmer, 1984)

$\mathrm{SL}(n,\mathbb{Z})$ *does not act on M^{n-1}, preserving volume.*

Theorem (Zimmer, 1984, 1991)

- *Action preserves meas'ble Riemannian metric.*

  — special case of cocycle superrigidity

- *Action factors through a compact group K* (*measurably*):

$$\begin{array}{ccc} \Gamma & \longrightarrow & K \\ \updownarrow & & \updownarrow \\ M & \overset{a.e.}{\cong} & X \end{array}$$

Conjecture (Zimmer, 1984)

$\mathrm{SL}(n,\mathbb{Z})$ *does not act on* M^{n-1}, *preserving volume.*

Theorem (Morris-Zimmer, 2012)

$\mathrm{SL}(n,\mathbb{Q})$ *does not act on* M^{n-1}, *preserving volume.*

($\mathrm{SL}(n,\mathbb{Q})$ is a **discrete** group)

Remark

Proved thm for many other simple alg'ic grps $\mathbf{G}(\mathbb{Q})$

(not just $\mathrm{SL}(n,\mathbb{Q})$)

Conjecture

Suffices to assume $\dim M < n^2 - 1 = \dim \mathrm{SL}(n,\mathbb{R})$.

Remark

Easy if $n < 3$, so we assume $n \geq 3$.

Theorem (Morris-Zimmer, 2012)

$\mathrm{SL}(n,\mathbb{Q})$ *does not act on* M^{n-1}, *preserving volume.*

$$\Gamma_m = \mathrm{SL}(n,\mathbb{Z}[1/m]) \subset \mathrm{SL}(n,\mathbb{Q}) = \bigcup_m \Gamma_m = \Gamma_\infty.$$

Theorem (Zimmer, 1991)

Γ_m*-action factors through a cpct grp* K_m *acting on* X_m (*measurably*)

$$\begin{tikzcd} \Gamma_m \arrow[r] \arrow[d, <->] & K_m \arrow[d, <->] \\ M \arrow[r, phantom, "\overset{a.e.}{\cong}"] & X_m \end{tikzcd}$$

[Peter-Weyl] $\quad K_m \subset \bigtimes_{i=1}^{\infty} \mathrm{SO}(N_i) \implies K_m$ is pro-Lie

Theorem (Margulis, 1974)

$\varphi : \Gamma_m \to$ *cpct Lie grp* $\mathrm{SO}(N_i) \implies \varphi(\Gamma)$ *is finite.*

$\therefore K_m$ is pro-finite $\overset{\text{wolog}}{=}$ pro-finite completion of Γ_m.

Γ_m-action factors through action of $\widehat{\Gamma}_m$ on M (a.e.)

$\Gamma_m = \mathrm{SL}(n,\mathbb{Z}[1/m]) \subset \mathrm{SL}(n,\mathbb{Q}) = \Gamma_\infty$.

Theorem (Congruence Subgroup Property)

$$\widehat{\Gamma}_1 = \bigtimes_p \mathrm{SL}(n,\mathbb{Z}_p), \qquad \widehat{\Gamma}_m = \bigtimes_{p \nmid m} \mathrm{SL}(n,\mathbb{Z}_p)$$

$\Gamma_1 \subset \Gamma_m \implies \widehat{\Gamma}_1 \to \widehat{\Gamma}_m$ with kernel $\bigtimes_{p \mid m} \mathrm{SL}(n,\mathbb{Z}_p)$

So $\bigtimes_{p \mid m} \mathrm{SL}(n,\mathbb{Z}_p)$ acts trivially on M (a.e.).

$\bigcup_m \bigtimes_{p \mid m} \mathrm{SL}(n,\mathbb{Z}_p)$ dense in $\bigtimes_p \mathrm{SL}(n,\mathbb{Z}_p) = \widehat{\Gamma}_1$

$\implies \widehat{\Gamma}_1$ acts trivially on M (a.e.)

$\implies \Gamma_1$ acts trivially on M.

Action of Γ_∞ has kernel, but $\Gamma_\infty = \mathrm{SL}(n,\mathbb{Q})$ simple. □

Theorem (Morris-Zimmer, 2012)

$\mathrm{SL}(n,\mathbb{Q})$ *does not act on* M^{n-1}, *preserving volume.*

Theorem (more general)

- $\mathbf{G}$ *is almost-simple* $\mathbb{Q}$*-group, satisfying* $(*)$, *and*
- $\nexists$ *homo* $\mathbf{G}(\mathbb{R})^\circ \to \mathrm{GL}(d;\mathbb{C})$.

$\implies \mathbf{G}(\mathbb{Q})$ *does not act on* M^d, *preserving volume.*

1. **Higher rank**: $\forall$ place v of $\mathbb{Q}$, $\mathbb{Q}_v$-rank(every simple factor of $\mathbf{G}(\mathbb{Q}_v)$) ≥ 2.
   $\implies$ *cocycle superrigidity* and *Kazhdan* (T)
2. **Congruence Subgroup Property** for large S-arithmetic subgroups of $\widetilde{\mathbf{G}}(\mathbb{Q})$.
   (OK unless $\mathbf{G}$ is anisotropic of type A_n, D_4, E_6.)
3. $\widetilde{\mathbf{G}}(\mathbb{Q})$ is almost simple (not really necessary).

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