Orbits of Cartan subgroups
on homogeneous spaces
(after George Tomanov and Barak Weiss)

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$G = \mathrm{SL}(n,\mathbb{R})$
$\quad = (\mathbb{R}$-pts of) Zar conn, reductive $\mathbb{Q}$-group

$\Gamma = \mathrm{SL}(n,\mathbb{Z})$
$\quad =$ arithmetic subgroup of G

$$A = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$
$\quad =$ Cartan subgroup of G
$\quad =$ maximal $\mathbb{R}$-split torus in G

A acts on G/Γ: $(a,[g]) \mapsto a[g]$.
$\quad [g] = g\Gamma$ for $g \in G$.

Ques. *Which orbit closures are homogeneous?*
(As in Ratner's thm for unipotent subgrps.)
¿ $\exists$ subgroup $L \supset A$, $\overline{A[g]} = L[g]$?

Today. Which orbits are **closed**.
I.e., when can we take $L = A$?

Eg. $G = \mathrm{SL}(2,\mathbb{R}) \qquad \Gamma = \mathrm{SL}(2,\mathbb{Z})$.

A-orbit = geodesic

$A[g]$ is closed
$$\Leftrightarrow \begin{cases} \text{periodic} & a^{\ell}[g] = [g],\ \exists \ell \in \mathbb{R}^+ \\ \quad\text{or} & \\ \text{divergent} & a^t[g] \to \infty \text{ as } t \to \pm\infty \end{cases}$$
$$\Leftrightarrow \begin{cases} g^{-1}Ag \cap \mathrm{SL}(2,\mathbb{Z}) \text{ is infinite} \\ \qquad\qquad\text{or} \\ \text{both } \mathbb{H}^2 \text{ endpts of } Ag \text{ are in } \mathbb{Q} \cup \{\infty\} \end{cases}$$
$$\Leftrightarrow \begin{cases} g^{-1}Ag \text{ is defined}/\mathbb{Q} \text{ and } \mathbb{Q}\text{-anisotropic} \\ \qquad\qquad\text{or} \\ g^{-1}Ag \text{ is defined}/\mathbb{Q} \text{ and } \mathbb{Q}\text{-split} \end{cases}$$
$$\Leftrightarrow g^{-1}Ag \text{ is defined}/\mathbb{Q}.$$

This generalizes:

Thm (Tomanov and Weiss, 2001).
$A[g]$ is closed $\Leftrightarrow$ $g^{-1}Ag$ is defined over $\mathbb{Q}$.
(More complicated for nonreductive groups.)

Thm. *$A[g]$ is closed $\Leftrightarrow$ $g^{-1}Ag$ is defined over $\mathbb{Q}$.*

Proof. ($\Leftarrow$) General fact:
$\quad H$ any $\mathbb{Q}$-torus of G
$\quad\quad \Rightarrow H$-orbit of $[e]$ is closed.

Therefore $(g^{-1}Ag)[e]$ is closed, so
$\quad A[g] = g(g^{-1}Ag)[e]$ is closed. $\square$

We only need to prove ($\Rightarrow$).

Special case:

Defn. Orbit of $[g]$ *divergent*: $a[g] \to \infty$ as $a \to \infty$.
$\quad \forall$ cpct $K \subset G/\Gamma$, $\exists$ cpct $C \subset A$,
$\quad\quad a \notin C \Rightarrow a[g] \notin K$.

Rem. Every divergent orbit is closed.
($a \mapsto a[g]$ is proper map, so closed $\to$ closed.)

Cor. *Orbit of $[g]$ is divergent $\Leftrightarrow$ $g^{-1}Ag$ is $\mathbb{Q}$-split.*

Thm. *$A[g]$ is closed $\Leftrightarrow$ $g^{-1}Ag$ is defined over $\mathbb{Q}$.*

Cor. *Orbit of $[g]$ is divergent $\Leftrightarrow$ $g^{-1}Ag$ is $\mathbb{Q}$-split.*

Proof of Thm from Cor. Let $T = g^{-1}Ag$.
T-orbit of $[e]$ is closed.
So $T[e] \cong T/(T \cap \Gamma)$.

R = Zariski closure of $T \cap \Gamma$
= $\mathbb{Q}$-anisotropic torus $\subset T$.
So $R/(T \cap \Gamma)$ is compact.

Let $\overline{G} = C_G(R)/R$ and $\overline{T} = T/R$.
- $\overline{G}$ is a connected, reductive $\mathbb{Q}$-group and
- $\overline{T}$ is a Cartan subgroup.

$\overline{T}$-orbit of $[e]$ is closed, with no stabilizer
$\Rightarrow$ divergent.

Cor tells us $\overline{T}$ is $\mathbb{Q}$-split.
In particular, it is defined over $\mathbb{Q}$ in $\overline{G}$.
Pull back to G: conclude T is defined over $\mathbb{Q}$. □

Cor. *Orbit of $[g]$ is divergent $\Leftrightarrow$ $g^{-1}Ag$ is $\mathbb{Q}$-split.*

We give a direct proof.

Special case (Margulis).
Assume $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$.

Lem (Mahler Compactness Criterion).
Orbit of $[g]$ is **not** *divergent* $\Leftrightarrow$
$\exists$ *neigh* Ω_0 *of* 0 *in* $\mathbb{R}^n$,
$\forall$ *cpct* $C \subset A$,
$\exists a \notin C$,
$ag\mathbb{Z}^n \cap \Omega_0 = \{0\}$.

An idea from Kazhdan-Margulis:

Lem. $\exists$ *neigh* Ω *of* 0, *finite* $F \subset A$, $c > 1$,
$\forall g \in G$, $\exists f \in F$,
$\forall v \in g\mathbb{Z}^n \cap \Omega$,
$\|fv\| \geq c\|v\|$.

Proof. Easy: $\forall$ subspace $V \subsetneq \mathbb{R}^n$, $\exists f \in A$, s.t.
$\|fv\| \geq c\|v\|$ for all $v \in V$.

Compactness of Grassman variety
$\Rightarrow$ $\exists$ finitely many f's to work for all V.

g unimodular $\Rightarrow$ $\bigl(\text{span of } (g\mathbb{Z}^n \cap \Omega)\bigr) \neq \mathbb{R}^n$. □

Lem. $\forall g \in G$, *s.t.* $g \notin A\,G_{\mathbb{Q}}$, ($G_{\mathbb{Q}} = \mathrm{SL}(n, \mathbb{Q})$)
$\forall$ *cpct* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $\forall$ *cpct* $C \subset A$,
$\exists a \in A \smallsetminus C$, $\forall v \in g\mathbb{Z}^n \smallsetminus 0$,
$v \in \Omega_1 \Rightarrow av \notin \Omega_2$.

Proof. Assumption on g:
some coord axis does not intersect $g\mathbb{Z}^n \smallsetminus 0$.
Say $g\mathbb{Z}^n \cap \{(0, 0, \ldots, 0, *)\} = \{0\}$.

Let $a^t = \begin{bmatrix} e^t & & & & \\ & e^t & & & \\ & & \ddots & & \\ & & & e^t & \\ & & & & e^{-(n-1)t} \end{bmatrix}$.

Then $a^t v \to \infty$ for all nonzero $v \in g\mathbb{Z}^n$.

So a large element of A takes the finitely many nonzero elements of $g\mathbb{Z}^n \cap \Omega$ far away. □

Cor. *Orbit of $[g]$ is divergent $\Rightarrow g^{-1}Ag$ is $\mathbb{Q}$-split.*

Proof. If $g \in A\,G_{\mathbb{Q}}$, then $g^{-1}Ag$ is $\mathbb{Q}$-split
(because A is $\mathbb{Q}$-split).

Thus, we assume $g \notin A\,G_{\mathbb{Q}}$
and show that the orbit is not divergent.

Mahler: we want $\Omega_0 \ni 0$, s.t.
$\forall$ cpct $C \subset A$, $\exists a \notin C$,
$ag\mathbb{Z}^n \cap \Omega_0 = \{0\}$.

Choose a Kazhdan-Margulis neighborhood Ω, with corresponding finite $F \subset A$ and $c > 1$.

$\Omega_0 =$ small ball around 0, s.t. $f^{\pm 1}v \in \Omega_0 \Rightarrow v \in \Omega$.

Given cpct $C \subset A$.

Using 2nd lemma, choose $a_0 \notin C$, s.t.
$0 \neq v \in g\mathbb{Z}^n$, $\exists c \in C$, $cv \in \Omega$
$\Rightarrow a_0 v \notin \Omega_0$.

If $a_0 g\mathbb{Z}^n \cap \Omega_0 = \{0\}$, we are done.

Otherwise, $\exists\ a_1 \in F$ that stretches $a_0 g\mathbb{Z}^n \cap \Omega$.

Continue stretching, until
$a_m \cdots a_1 a_0 g\mathbb{Z}^n \cap \Omega_0 = \{0\}$.

We are done, unless $a_m \cdots a_1 a_0 \in C$.

Since the intersection $\neq 0$ on the previous step,
$\exists\ v \in g\mathbb{Z}^n \smallsetminus \{0\}$, s.t. $a_{m-1} \cdots a_1 a_0 v \in \Omega_0$.
We must have $a_0 v \in \Omega_0$ and
$a_m \cdots a_1 a_0 v \in \Omega$.

This contradicts the choice of a_0. $\square$

For general G, Tomanov and Weiss generalize the three lemmas, using the adjoint representation.

Lem (Mahler Compactness Criterion).
Orbit of $[g]$ is **not** *divergent $\Leftrightarrow$*
$\exists$ neigh Ω_0 of 0 in $\mathbb{R}^n$,
$\forall$ cpct $C \subset A$,
$\exists a \notin C$,
$ag\mathbb{Z}^n \cap \Omega_0 = \{0\}$.

Use Fundamental Domain $\cup_q K S_t M q$ to show similar, with Lie algebra $\mathfrak{g}$ in place of $\mathbb{R}^n$:

Lem. *Orbit of $[g]$ is* **not** *divergent $\Leftrightarrow$*
$\exists$ neighborhood Ω_0 of 0 in $\mathfrak{g}$,
$\forall$ cpct $C \subset A$,
$\exists a \notin C$,
$\mathrm{Ad}(ag)\mathfrak{g}_{\mathbb{Z}} \cap \Omega \not\supset$ *horospherical set.*

Defn. Horospherical set: basis of subspace conjugate to Lie algebra of unipotent radical of a maximal parabolic $\mathbb{Q}$-subgroup.

Lem. *$\exists$ neigh Ω of 0, finite $F \subset A$, $c > 1$,*
$\forall g \in G$, $\exists f \in F$, $\quad \forall v \in g\mathbb{Z}^n \cap \Omega$,
$\|fv\| \geq c\|v\|$.

Just replace $g\mathbb{Z}^n$ with $(\mathrm{Ad} g)\mathfrak{g}_{\mathbb{Z}}$.

Proof. $A \subset P =$ minimal parabolic $\mathbb{R}$-subgroup.

Kazhdan-Margulis:
Subalgebra gen'd by $\big((\mathrm{Ad} g)\mathfrak{g}_{\mathbb{Z}}\big) \cap \Omega$ is unipotent
$\subset \mathfrak{q}_{\mathrm{unip}}$, $\exists$ minimal parabolic $\mathbb{R}$-subgroup Q.

Suffices to show: $\exists w \in W$ (Weyl group), s.t.
$wQ_{\mathrm{unip}}w^{-1} \cap P^- = e$.

$\exists g \in G$, $Q = g^{-1}Pg$.
Bruhat: $g = bcw$ with $b \in P$, $c \in P^-$, $w \in W$.

Then $wQ_{\mathrm{unip}}w^{-1} = wg^{-1}P_{\mathrm{unip}}gw^{-1} = c^{-1}P_{\mathrm{unip}}c$ does not intersect P^-.
(Since $U \cap P^- = e$, and $c \in N_G(P^-)$.)

Now assume G is $\mathbb{Q}$-simple (wolog).

Lem. *$\forall g \in G$, s.t. $g \notin A\, G_{\mathbb{Q}}$,*
$\forall$ cpct $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $\forall$ cpct $C \subset A$,
$\exists a \in A \smallsetminus C$, $\forall v \in g\mathbb{Z}^n \smallsetminus 0$,
$v \in \Omega_1 \Rightarrow av \notin \Omega_2$.

Let $S \subset A$ be a maximal $\mathbb{Q}$-split torus of G.
(Maybe have to replace A by a conjugate.)

Lem C. *$\forall g \in G$, s.t. $g \notin C_G(S)\, G_{\mathbb{Q}}$,*
and $(g^{-1}Sg) \cap \Gamma$ is finite,
$\forall$ cpct $\Omega_1, \Omega_2 \subset \mathfrak{g}$, $\forall$ $C \subset A$,
$\exists a \in S \smallsetminus C$,
$\forall$ horospherical subset H of $(\mathrm{Ad} g)\mathfrak{g}_{\mathbb{Z}}$,
$H \subset \Omega_1 \Rightarrow (\mathrm{Ad} a)H \not\subset \Omega_2$.

Rem. Because all maximal $\mathbb{Q}$-split tori are conjugate under $G_{\mathbb{Q}}$ the assumption that $g \notin C_G(S)\, G_{\mathbb{Q}}$, is equivalent to saying that $g^{-1}Sg$ is not $\mathbb{Q}$-split.

Need way to show $g^{-1}Sg$ is $\mathbb{Q}$-split:

Lem.
- *$g \in G$,*
- *$A \subset P$ = minimal parabolic $\mathbb{Q}$-subgroup,*
- *$\forall w \in W_{\mathbb{Q}}$, $\exists$ proper $Q^w \supset w^{-1}Pw$*
  s.t. $g^{-1}Q^w g$ is defined over $\mathbb{Q}$,

$\Rightarrow g^{-1}Sg$ is defined over $\mathbb{Q}$.

Proof. $g^{-1}Q^w g$ def'd over $\mathbb{Q} \Rightarrow \forall \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$,
- $\sigma(Q^w) \supset \sigma(w^{-1}Pw) = w^{-1}Pw$, and
- $g^{-1}Q^w g = \sigma(g^{-1}Q^w g) = \sigma(g^{-1})\, \sigma(Q^w)\sigma(g)$

Q^w, $\sigma(Q^w)$ are conj, and contain same Borel,
so $Q^w = \sigma(Q^w)$.
Thus, $\sigma(g)g^{-1} \in \cap_{w \in W} Q^w = C_G(S)$.

So $g^{-1}Sg$ is defined over $\mathbb{Q}$. □

Proof of Lem C. $A \subset P$ = min'l parab $\mathbb{Q}$-subgrp.

For $w \in W_{\mathbb{Q}}$ (Weyl group):
Assume $(\mathrm{Ad} w)\mathfrak{p} \cap (\mathrm{Ad} g)\mathfrak{g}_{\mathbb{Z}} \supset$ horo subset H^w
(else easy).

Let $Q^w = N_G(\text{span of } H^w)$.

$$g^{-1}Q^w g = N_G\big(\text{span of } (\mathrm{Ad} g^{-1})H^w\big)$$
$$= N_G(\text{span of subset of } \mathfrak{g}_{\mathbb{Z}})$$
$\Rightarrow g^{-1}Q^w g$ is defined over $\mathbb{Q}$.

$\mathfrak{q}^w_{\text{unip}} = (\text{span of } H^w) \subset w\mathfrak{p}w^{-1}$,
so $wPw^{-1} \subset Q^w$.

Thus, the lemma implies $g^{-1}Sg$ is defined over $\mathbb{Q}$.

Since (by assumption) $(g^{-1}Sg) \cap \Gamma$ is finite, we conclude that $g^{-1}Sg$ is $\mathbb{Q}$-split.

Lem.
- *P = minimal parabolic $\mathbb{Q}$-subgroup of G,*
- *S = maximal $\mathbb{Q}$-split torus of P,*
- *$\mathbb{Q}$-subgrp $Q^w \supset w^{-1}Pw$, for each $w \in W_{\mathbb{Q}}$,*

$\Rightarrow \cap_{w \in W} Q^w = C_G(S)$.

Proof. Following page: $\cap_{w \in W} Q^w$ is connected.
So consider only the Lie algebra.

Note that this is a sum of root spaces $\mathfrak{g}_\alpha$.

α must be orthogonal to some root in each set Δ of simple $\mathbb{Q}$-roots.

This is impossible: all roots of same length are conjugate under $W_{\mathbb{Q}}$.

Maximal root not orthogonal to any simple root. □

Lem. *Let $P_1, \ldots, P_n$ be parabolic subgroups of G. If $P_1 \cap \cdots \cap P_n$ contains a maximal torus of G, then*

- *$P_1 \cap \cdots \cap P_n$ is connected, and*
- *$\exists$ unipotent subgroup U of G,*
  normalized by $P_1 \cap \cdots \cap P_n$,
  such that $(P_1 \cap \cdots \cap P_n)U$ is parabolic.

Prove by induction on n from:

Lem. *P and Q parabolic $\Rightarrow$*

- *$P \cap Q$ connected, and*
- *$(P \cap Q)P_{\mathrm{unip}}$ is parabolic.*

Proof. Parabs are conn, so suffices to show latter.

$T =$ max'l torus contained in both P and Q.

For root α, show $(\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{p}_{\mathrm{unip}} \supset$ either $\mathfrak{g}_\alpha$ or $\mathfrak{g}_{-\alpha}$.

Neither $\mathfrak{g}_\alpha$ nor $\mathfrak{g}_{-\alpha}$ is in $\mathfrak{p}_{\mathrm{unip}} \Rightarrow$ both are in $\mathfrak{p}$,
and one or the other is in $\mathfrak{q}$. □

Cor. *Suppose*

- *P is a minimal parabolic $\mathbb{Q}$-subgroup,*
- *Q is a conjugate of a parabolic $\mathbb{Q}$-subgroup, and*
- *$Q_u \subset P$.*

$\Rightarrow P \subset Q$, and Q is defined over $\mathbb{Q}$.

Proof. $\exists Q' \supset P$, s.t.

- Q' is conjugate to Q, and
- Q' is a $\mathbb{Q}$-subgroup.

Preceding page: $(P \cap Q)Q_{\mathrm{unip}}$ is parabolic.
Contained in both P and Q,
so contained in both Q' and Q.

Therefore $Q' = Q$. □

References

G. Tomanov and B. Weiss: Closed orbits for actions of maximal tori on homogeneous spaces, *Duke Math. J.* (to appear).
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