What is a superrigid subgroup?

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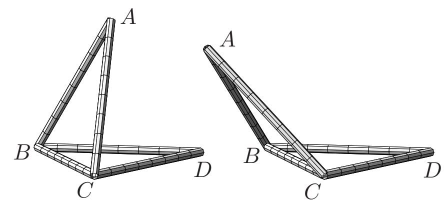
Abstract. In combinatorial geometry (and engineering), it is important to know that certain scaffold-like geometric structures are rigid. (They will not collapse, and, in fact, have enough bracing that they cannot be deformed at all.) Replacing the geometric structure with an algebraic structure (namely, a group) leads to the following question: given a homomorphism that is defined on the elements of a subgroup, is it possible to extrapolate the homomorphism to the rest of the elements of the group? It is fairly obvious that every additive homomorphism from the group $\mathbb{Z}$ of integers to the real line $\mathbb{R}$ can be extended to a homomorphism that is defined on all of $\mathbb{R}$, and we will see some other examples.

Combinatorial superrigidity

Example (Two joined triangles)

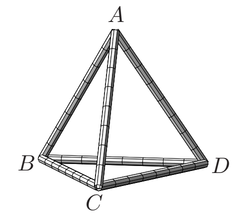
A——B
A——C
B——C
B——D
C——D



This is not rigid. I.e., it can be deformed (a “hinge").

Example (Tetrahedron)

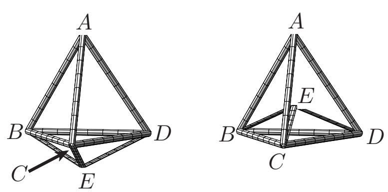
A——B
A——C
A——D
B——C
B——D
C——D



This is **rigid** (cannot be deformed).

Example (Add a small tetrahedron)

A——B B—E
A——C C—E
A——D D—E
B——C
B——D
C——D

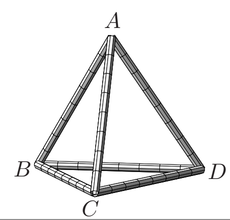


This is rigid.

However, it is not **superrigid**:
if it is taken apart, it can be reassembled incorrectly.

Example (Tetrahedron)

A——B
A——C
A——D
B——C
B——D
C——D



This **is** superrigid: the combinatorial description determines the geometric structure.

Combinatorial superrigidity

Make a copy of the object,
according to the combinatorial rules.
The copy is the exact same shape as the original.

This talk: analogue in group theory.

The analogy

Combinatorial superrigidity

Make a copy of the object,
according to the combinatorial rules.
The copy is the exact same shape as the original.

Maybe not exactly the same object:
may be *translated* from the original position,
may be *rotated* from the original position,
may be *reflected* from the original position.

These are trivial/obvious modifications:
translations, rotations, & reflections are symmetries of the whole universe (Euclidean space $\mathbb{R}^3$).

Combinatorial superrigidity

Make a copy of the object,
according to the combinatorial rules.
Can get same result by keeping the original object and moving the whole universe to a new position.

“If the object can be moved somewhere,
then the whole universe can be moved there."

Group-theoretic superrigidity

Let H be a subgrp of a grp G, and make a copy of H.
Same copy of H can be obtained by moving all of G.

Every homomorphism $\varphi\colon H \to \boxed{?}$
extends to homomorphism $\hat{\varphi}\colon G \to \boxed{?}$.

Group-theoretic superrigidity

Example

Group homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{R}^d$
(i.e., $\varphi(m+n) = \varphi(m) + \varphi(n)$)
$\Rightarrow$ φ extends to homomorphism $\hat{\varphi}\colon \mathbb{R} \to \mathbb{R}^d$.
Namely, define $\hat{\varphi}(x) = x \cdot \varphi(1)$.

Check: $\hat{\varphi}(n) = \varphi(n)$, $\hat{\varphi}(x+y) = \hat{\varphi}(x) + \hat{\varphi}(y)$,
$\hat{\varphi}$ is continuous (only allow continuous homomorphisms)

Any homo into $\mathbb{R}^d$ can be thought of as a homo into a *matrix group*:

$$\mathbb{R}^3 \cong \begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Group Representation Theory:

Study homomorphisms into *Matrix Groups*.
$\mathrm{GL}_d(\mathbb{C}) = \{\text{invertible } d \times d \text{ matrices over } \mathbb{C}\}$

Example

Group homomorphism $\varphi\colon \mathbb{Z} \to \mathrm{GL}_d(\mathbb{R})$
(i.e., $\varphi(m+n) = \varphi(m) \cdot \varphi(n)$)
$\not\Rightarrow$ extends to continuous homo $\hat{\varphi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$.

Proof by contradiction.

Spse $\exists\, \hat{\varphi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$, s.t. $\hat{\varphi}(n) = \varphi(n)$, $\forall n \in \mathbb{Z}$.
$\hat{\varphi}(0) = I \quad \Rightarrow \det(\hat{\varphi}(0)) = 1 > 0$
$\mathbb{R}$ connected $\Rightarrow \hat{\varphi}(\mathbb{R})$ conn $\Rightarrow \det(\hat{\varphi}(\mathbb{R}))$ conn $\not\ni 0$
$\Rightarrow \det(\hat{\varphi}(1)) \in \det(\hat{\varphi}(\mathbb{R})) > 0 \quad \rightarrow\leftarrow$

Example

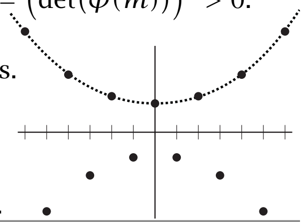
Group homo $\varphi\colon \mathbb{Z} \to \mathrm{GL}_d(\mathbb{R})$
$\not\Rightarrow$ extension $\hat{\varphi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$. $\left(\det(\varphi(1)) \overset{?}{<} 0\right)$

$$\det(\varphi(\text{even})) = \det(\varphi(2m)) = \det(\varphi(m+m))$$
$$= \det(\varphi(m)\cdot\varphi(m)) = \left(\det(\varphi(m))\right)^2 > 0.$$

May have to ignore odd #s:
restrict attention to even #s.

Analogously, may need to
restrict to multiples of 3
(or 4 or 5 or ...).

Restrict attn to mults of N.



Restrict attention to a finite-index subgroup.

Superrigid subgroups

Proposition

Group homomorphism $\varphi\colon \mathbb{Z}^k \to \mathrm{GL}_d(\mathbb{R})$
$\Rightarrow$ φ virtually extends to homo $\hat{\varphi}\colon \mathbb{R}^k \to \mathrm{GL}_d(\mathbb{R})$.

"Homomorphisms defined on $\mathbb{Z}^k$
virtually extend to be defined on $\mathbb{R}^k$."
This means $\mathbb{Z}^k$ is *superrigid* in $\mathbb{R}^k$.

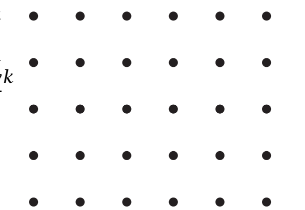
Generalize to nonabelian groups.

$\mathbb{Z}^k$ is superrigid in $\mathbb{R}^k$

$\mathbb{Z}^k$ is a (uniform) *lattice* in $\mathbb{R}^k$:

- $\mathbb{R}^k$ is a connected group ("Lie group")
- $\mathbb{Z}^k$ is a discrete subgroup
- all of $\mathbb{R}^k$ is within a bounded distance of $\mathbb{Z}^k$
  $\exists C,\ \forall x \in \mathbb{R}^k,\ \exists m \in \mathbb{Z}^k,\quad d(x,m) < C$

All of $\mathbb{R}^k$
is within
$\sqrt{k}/2$ of $\mathbb{Z}^k$



Definition.
H is a *lattice* in G:
replace $\mathbb{Z}^k$ with H,
and $\mathbb{R}^k$ with G.

Margulis Superrigidity Theorem ~1973

Every lattice in $\mathrm{SL}_n(\mathbb{R})$ *is superrigid* *if $n \geq 3$.*
(up to bounded error — i.e., modulo compact subgroup)

Example: Lattice in $\mathrm{SL}_2(\mathbb{R})$

$$\mathbb{H}_3 = \left\{ \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \;\middle|\; \begin{array}{l} ij = k = -ji, \\ i^2 = -1,\ j^2 = 3 \end{array} \right\} \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$$

$\mathrm{SL}_2(\mathbb{R}) \cong \{\, g \in \mathbb{H}_3 \mid g\overline{g} = 1 \,\}$

$H = (\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k) \cap \mathrm{SL}_2(\mathbb{R})$ is lattice. $\left(\begin{array}{c}\text{not}\\ \text{superrig}\end{array}\right)$

Superrigidity Theorem: all lattices in $\mathrm{SL}_n(\mathbb{R})$ $(n \geq 3)$
come from similar constructions, using $\mathbb{Z}$-points.
All lattices are "*arithmetic*."

Margulis Superrigidity Theorem ~1973

Every lattice in $\mathrm{SL}_n(\mathbb{R})$ *is superrigid* *if $n \geq 3$.*
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Margulis Arithmeticity Theorem ~1973

Every lattice in $\mathrm{SL}_n(\mathbb{R})$ is "arithmetic" if $n \geq 3$.

Provides a list of all the lattices in $\mathrm{SL}_n(\mathbb{R})$:
all the "periodic tilings" or "crystals"

Can replace $\mathrm{SL}_n(\mathbb{R})$ with any simple Lie grp except $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$.

Key. Show matrix entries of H are in $\mathbb{Z}$
for some embedding $\underset{\times\text{compact}}{\mathrm{SL}_n(\mathbb{R})} \hookrightarrow \mathrm{SL}_N(\mathbb{R})$.

Why superrigidity implies arithmeticity

Prop. H = superrigid lattice in $\mathrm{SL}_n(\mathbb{R})$
$\Rightarrow$ ~ every matrix entry is in $\mathbb{Z}$.

Step 1. Show $h_{i,j}$ is not transcendental

If it is, then $\exists$ field auto φ of $\mathbb{C}$ with $\varphi(h_{i,j}) = ???$

Define $\tilde{\varphi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}$.

So $\tilde{\varphi}\colon H \to \mathrm{GL}_n(\mathbb{C})$ is a group homomorphism.

Superrigidity: $\tilde{\varphi}$ extends to $\hat{\varphi}\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{C})$.

There are uncountably many different φ's, but
$\mathrm{SL}_n(\mathbb{R})$ has $< \infty$ n-dim'l rep'ns (up to change of basis). $\rightarrow\leftarrow$

H is a superrigid lattice in $\mathrm{SL}_n(\mathbb{R})$, and
1. every matrix entry is algebraic (e.g., in $\mathbb{Q}$).

Step 2. Show no denominators

Actually, show denominators are bounded.
(Then finite-index subgrp has no denoms.)

Fact. H is generated by finitely many matrices.

Only finitely many primes appear in denoms,
so just show each prime occurs to bounded power.

Theorem ("p-adic superrigidity" [Margulis])

H latt in $\mathrm{SL}_n(\mathbb{R})$ $(n \geq 3)$, *$\varphi\colon H \to \mathrm{SL}_k(\mathbb{Q}_p)$ grp homo*
$\Rightarrow$ $\varphi(H)$ has compact closure.
I.e., $\exists m$, no matrix in $\varphi(H)$ has p^m in denom.

Bounded generation

Note: Invertible matrix $\rightsquigarrow I$ by row operations.
$g \in \mathrm{SL}_n(\mathbb{Z}) \rightsquigarrow I$ by integer ($\mathbb{Z}$) row ops.

$$\begin{bmatrix} 9 & 20 \\ 49 & 109 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem (Carter-Keller 1983)

Matrix in $\mathrm{SL}_3(\mathbb{Z})$ *needs bdd # of $\mathbb{Z}$ row ops.* (< 50)

But no bound on # for mats in $\mathrm{SL}_2(\mathbb{Z})$.

This implies $\mathrm{SL}_3(\mathbb{Z})$ is superrigid (and Cong Subgrp Prop).

Converse? Is every lattice in $\mathrm{SL}_3(\mathbb{R})$ *bddly gen'd*?
¿ $\langle h_1 \rangle \langle h_2 \rangle \cdots \langle h_r \rangle = H$?

- A. K. Dewdney, Mathematical Recreations: the theory of rigidity, or how to brace yourself against unlikely accidents, *Scientific American*, May 1991, 126–128. `http://triamant.info/html_e/history.html`
- J. Graver, B. Servatius, & H. Servatius, *Combinatorial Rigidity*, Amer. Math. Soc., Providence, RI, 1993. MR 1251062
- D. W. Morris: What is a superrigid subgroup?, in: *Communicating Mathematics*, Amer. Math. Soc., Providence, RI, 2009, pp. 189–206. MR 2513447 `http://arxiv.org/abs/0712.2299`