$SL_3(\mathbb{Z})$ **cannot act continuously on the circle**

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Circles are very closely related to lines:

- The universal cover of a circle is a line.
- Removing a point from a circle leaves a line.

Thm (Ghys). Let Γ be an irreducible lattice in a connected semisimple Lie group G with finite center. Assume:

1) $\mathbb{R}$-rank$G \geq 2$.

2) No quotient of G is $\cong PSL_2(\mathbb{R})$.

3) Γ cannot act faithfully on a finite union of lines.

Then Γ cannot act faithfully on a circle.

In fact, Ghys proved (without using (3)) that any action of Γ on a circle must have a finite orbit. Removing the finitely many points in this orbit leaves a finite union of line segments.

Therefore, it suffices to discuss actions on a line.

In fact, we consider actions on a line *segment* $I = [-1, 1]$.

All actions are assumed to be by homeomorphisms.

Also (for simplicity), all actions are assumed to be orientation-preserving

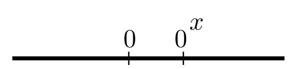
- I.e., if $a < b$, then $\gamma(a) < \gamma(b)$.
- In particular, $\gamma(-1) = -1$ and $\gamma(1) = 1$

Proposition (well known). I has no (nontrivial) orientation-preserving homeomorphism of finite order.

Proof. Let γ be a homeomorphism of $[-1, 1]$.

Some point is moved by γ. It might as well be 0.

Suppose $0 < 0^{\gamma}$.



Then $0^{\gamma} < (0^{\gamma})^{\gamma}$ (bcs γ is orient-pres).

Hence $0 < 0^{\gamma^2}$.

Then $0^{\gamma} < (0^{\gamma^2})^{\gamma}$. I.e., $0^{\gamma^3} > 0^{\gamma}$. Hence $0 < 0^{\gamma^3}$.

...

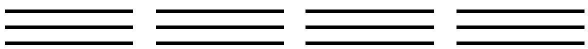
No power of γ fixes 0.

No power of γ is trivial, so γ has infinite order.

Cor. $\mathrm{SL}_2(\mathbb{Z})$ cannot act nontrivially on I.

Proof. $\mathrm{SL}_2(\mathbb{Z})$ is generated by elements of finite order.

Remark. $\mathrm{SL}_2(\mathbb{Z})$ **can** act faithfully on a finite union of line segments.



I.e. A finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ can act faithfully on a line segment.

Proof. $\mathrm{SL}_2(\mathbb{Z})$ has a free subgroup of finite index.

Thm (Witte). Let Γ be a finite-index subgroup of $\mathrm{SL}_3(\mathbb{Z})$.
Then Γ **cannot** act faithfully on I.

Cor. Let Γ be a finite-index subgroup of $\mathrm{SL}_3(\mathbb{Z})$. Then Γ cannot act (nontrivially) on I.

Proof. The kernel of the action is nontrivial.

Fact. Every normal subgrp of Γ is either finite or finite-index.
[E.g., a theorem of Margulis on normal subgroups of lattices.]

We conclude that the Γ-action factors through an action of a finite group.

No finite group can act on I.

Let Γ be a finite-index subgroup of $\mathrm{SL}_3(\mathbb{Z})$.
We now show that Γ cannot act faithfully on I.

Recall the proof that a line segment has no symmetries of finite order.

"Suppose 0^γ is to the right of 0."

For any symmetry γ of I, we say that

- γ is **positive** if $0^\gamma > 0$
- γ is **negative** if $0^\gamma < 0$

[Actually: Well-order I. Consider $p^\gamma >< p$, where p is the first point that is moved by γ.]

Remark. Every nontrivial symmetry of a line segment is either positive or negative (not both).

- The product of positives is positive.
- The inverse of a positive is negative.

Thm (well known). The group of orientation-preserving homeomorphisms of I is **right orderable**.

(Define $\phi < \psi$ if $\phi\psi^{-1}$ is negative.
Then $\phi < \psi \Leftrightarrow \phi\sigma < \psi\sigma$.)

Cor. Any group that acts faithfully on I must be right orderable.

Thm. $\mathrm{SL}_3(\mathbb{Z})$ is not right orderable.

(Nor subgroups of finite index [Witte].)

Impossible to do:

- Every nonidentity element is either positive or negative (but not both).
- The product of positives is positive.
- The inverse of a positive is negative.

$\mathrm{SL}_3(\mathbb{Z})$ contains the discrete Heisenberg group

$$H_{\mathbb{Z}} = \begin{bmatrix}1&*&*\\0&1&*\\0&0&1\end{bmatrix}, \qquad x = \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$y = \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, \qquad z = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$

$$xz = zx,\ yz = zy,\ xy = yxz$$

We may assume that x, y, z positive.

Key Step. Either $x \gg z$ or $y \gg z$.

Define $u \gg v$ if $u > v^r$ for all $r \in \mathbb{Z}$.

Proof. Suppose $x \not\gg z$ and $y \not\gg z$, so $x, y < z^r$.

$$x^a y^b = y^b x^a z^{ab}$$

$$(x^{-n}y^{-n})(x^n y^n) = (x^{-n}y^{-n})(y^n x^n z^{n^2}) = z^{n^2}$$

$$(x^{-n}y^{-n}x^n y^n)(z^{-2rn}) = x^{-n}y^{-n}(yz^{-r})^n(xz^{-r})^n$$

$< e$ so $x^{-n}y^{-n}x^n y^n < z^{2rn}$.

Therefore, $z^{n^2} < z^{2rn}$. $\rightarrow\leftarrow$

We actually have 6 copies of $H_{\mathbb{Z}}$ in $\mathrm{SL}_3(\mathbb{Z})$:

$$\mathrm{SL}_3(\mathbb{Z}) = \begin{bmatrix}*&1&2\\4&*&3\\5&6&*\end{bmatrix}$$

$$1,2,3 \quad 2,3,4 \quad 3,4,5$$

$$4,5,6 \quad 5,6,1 \quad 6,1,2$$

We have seen that $1 \gg 2$ or $3 \gg 2$.
Assume that $3 \gg 2$.

$2,3,4$: because $2 \not\gg 3$, we have $4 \gg 3$.

$3,4,5$: because $3 \not\gg 4$, we have $5 \gg 4$.

$4,5,6$: because $4 \not\gg 5$, we have $6 \gg 5$.

$5,6,1$: because $5 \not\gg 6$, we have $1 \gg 6$.

$6,1,2$: because $6 \not\gg 1$, we have $2 \gg 1$.

Therefore, $2 \gg 2$ — a contradiction.

A conceptual version of the proof.

Let Φ be the root system of $\mathrm{SL}_3(\mathbb{Q})$.

The root system of type A_2

$\alpha_i = \alpha_{i-1} + \alpha_{i+1}$ for $i = 1, 2, \ldots, 6$.

Key Step. If $\alpha, \beta \in \Phi$, and $\alpha + \beta \in \Phi$, then $\alpha + \beta \ll \alpha$ or $\alpha + \beta \ll \beta$.

$\alpha_1 \ll \alpha_1$ — a contradiction.

A similar proof works for $\mathrm{Sp}(4, \mathbb{Z})$.

The root system of type B_2

Either $\alpha_1 \ll \alpha_2 \ll \alpha_3$ or $\alpha_1 \ll \alpha_8 \ll \alpha_7$.
We have $\alpha_1 \ll \alpha_1$ — a contradiction.

Thm (Witte). Let Γ be an irreducible lattice in a connected, semisimple Lie group G with finite center. If Γ contains a finite-index subgroup of either $\mathrm{SL}_3(\mathbb{Z})$ or $\mathrm{Sp}(4, \mathbb{Z})$, then Γ cannot act nontrivially on a line.

Equivalent formulation: "If Γ is an arithmetic subgroup of a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least two, then …"

References

E. Ghys: Actions de réseaux sur le cercle (preprint).

D. Witte: Arithmetic groups of higher $\mathbb{Q}$-rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.* 122 (1994) 333–340.