A LATTICE WITH NO TORSION-FREE SUBGROUP OF FINITE INDEX (AFTER P. DELIGNE)

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Statement of the results

We discuss only a simple special case of Deligne's results.

Proposition (Deligne). *The inverse image of* $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$ *in the universal cover of* $\mathrm{Sp}(2n,\mathbb{R})$ *is not residually finite.*

Remark. In fact, Deligne showed that the intersection of all the subgroups of finite index is precisely $2\,\pi_1(\mathrm{Sp}(2n,\mathbb{R}))$.

Here is an equivalent formulation of the result that does not require infinite extensions. It is well known that the fundamental group of $G_{\mathbb{R}} = \mathrm{Sp}(2n,\mathbb{R})$ is $\mathbb{Z}$. Thus, for any finite cyclic group Z, there is a (unique) connected covering group $\widetilde{G_{\mathbb{R}}}$ of $G_{\mathbb{R}}$ with covering group Z. The inverse image $\widetilde{\Gamma}$ of $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$ in $\widetilde{G_{\mathbb{R}}}$ is a lattice in $\widetilde{G_{\mathbb{R}}}$.

Proposition (Deligne). *If* $\#Z > 2$, *then* $\widetilde{\Gamma}$ *has no torsion-free subgroup of finite index.*

Proof

We prove the contrapositive: assuming that $\widetilde{\Gamma}$ has a torsion-free subgroup of finite index (or, equivalently, $\widetilde{\Gamma}$ is residually finite), we will show $\#Z \le 2$.

It is known that Γ has the Congruence Subgroup Property. This means that the profinite completion of Γ is

$$\widehat{\Gamma} \cong \bigtimes_{p \text{ prime}} \mathrm{Sp}(2n,\mathbb{Z}_p),$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Since $\widetilde{\Gamma}$ is residually finite, the profinite completion of $\widetilde{\Gamma}$ is a central extension of $\widehat{\Gamma}$ by Z:

$$e \to Z \to \widehat{\widetilde{\Gamma}} \to \widehat{\Gamma} \to e.$$

Now, we will obtain an analogous result that replaces the arithmetic group $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$ with the algebraic group $G_{\mathbb{Q}} = \mathrm{Sp}(2n,\mathbb{Q})$. We define a topology on $G_{\mathbb{Q}}$ by declaring the subgroups of finite index in Γ to be the basic open sets containing e. (And the translates of these subgroups are the basic open sets at any other point. Since $G_{\mathbb{Q}}$ commensurabilizes Γ, it does not matter whether we translate on the left or on the right — the same topology is obtained.) Then we can complete $G_{\mathbb{Q}}$ to obtain a totally disconnected group $\widehat{G_{\mathbb{Q}}}$ that contains $\hat{\Gamma}$ as a compact open subgroup. Indeed, from the description of $\hat{\Gamma}$, we see that

$$\widehat{G_{\mathbb{Q}}} = \overset{\circ}{\bigtimes_{p \text{ prime}}} \operatorname{Sp}(2n, \mathbb{Q}_p),$$

where $\overset{\circ}{\times}$ represents the restricted (or "adelic") product in which all but finitely many coordinates g_p are required to be in the compact group $\operatorname{Sp}(2n, \mathbb{Z}_p)$. This is a locally compact group.

Note. The direct product $G_{\mathbb{R}} \times \widehat{G_{\mathbb{Q}}}$ is the adelic group $G_{\mathbb{A}} = \operatorname{Sp}(2n, \mathbb{A})$.

Similarly, we complete $\widetilde{G_{\mathbb{Q}}}$ with respect to the topology defined by the finite-index subgroups of $\tilde{\Gamma}$, so $\hat{\tilde{\Gamma}}$ is a compact open subgroup of $\widehat{\widetilde{G_{\mathbb{Q}}}}$.

We have a central extension:

$$e \to Z \to \widehat{\widetilde{G_{\mathbb{Q}}}} \to \widehat{G_{\mathbb{Q}}} \to e.$$

Taking the product of this with the central extension

$$e \to Z \to \widetilde{G_{\mathbb{R}}} \to G_{\mathbb{R}} \to e$$

yields a central extension

$$e \to Z \times Z \to \widetilde{G_{\mathbb{R}}} \times \widehat{\widetilde{G_{\mathbb{Q}}}} \to G_{\mathbb{A}} \to e.$$

Now, let

$$\widetilde{G_{\mathbb{A}}} = \frac{\widetilde{G_{\mathbb{R}}} \times \widehat{\widetilde{G_{\mathbb{Q}}}}}{\{\, (z,z) \mid z \in Z \,\}},$$

so we have a central extension

$$(*) \qquad e \to Z \to \widetilde{G_{\mathbb{A}}} \to G_{\mathbb{A}} \to e.$$

Note that the diagonal embedding

$$\widetilde{G_{\mathbb{Q}}} \to \widetilde{G_{\mathbb{R}}} \times \widehat{\widetilde{G_{\mathbb{Q}}}}$$

factors through to a well-defined embedding

$$G_{\mathbb{Q}} \to \widetilde{G_{\mathbb{A}}}.$$

Hence, the extension $(*)$ splits over $G_{\mathbb{Q}}$.

Now, the following theorem of C. C. Moore immediately implies that $\#Z \le 2$.

Theorem (Moore). *There is a "universal" topological central extension*

$$e \to \mathbb{Z}/2\mathbb{Z} \to E \to G_{\mathbb{A}} \to e$$

that splits over $G_{\mathbb{Q}}$, such that if

$$e \to Z \to \widetilde{G_{\mathbb{A}}} \to G_{\mathbb{A}} \to e$$

is any topological central extension that splits over $G_{\mathbb{Q}}$, with Z discrete, then there is a commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & E & \longrightarrow & G_{\mathbb{A}} & \longrightarrow & e \\
\| & & \downarrow & & \downarrow & & \| & & \| \\
e & \longrightarrow & Z & \longrightarrow & \widetilde{G_{\mathbb{A}}} & \longrightarrow & G_{\mathbb{A}} & \longrightarrow & e
\end{array}$$

In our case, we know that the map $E \to \widetilde{G_{\mathbb{A}}}$ is surjective, because $\widetilde{G_{\mathbb{R}}}$ is perfect and contains Z. Therefore Z is a quotient of $\mathbb{Z}/2\mathbb{Z}$, so $\#Z \le 2$, as desired.

Discussion of Moore's Theorem

It is well known that the fundamental group of the real Lie group $\mathrm{Sp}(2n,\mathbb{R})$ is $\mathbb{Z}$, so the universal cover is a central extension

$$e \to \mathbb{Z} \to \widetilde{\mathrm{Sp}(2n,\mathbb{R})} \to \mathrm{Sp}(2n,\mathbb{R}) \to e.$$

Deodhar (with details completed by Deligne) calculated an analogous "fundamental group" of the p-adic Lie group $\mathrm{Sp}(2n,\mathbb{Q}_p)$ (and of more general "quasi-split" p-adic groups).

Theorem (Deodhar, Deligne)**.** *For every prime p, there is a universal central extension $\widetilde{G_p}$ of* $\mathrm{Sp}(2n,\mathbb{Q}_p)$ *with discrete kernel. Furthermore, the kernel of the universal extension is the group μ_p of all roots of unity in $\mathbb{Q}_p^\times$.*

For almost every p, the extension $\widetilde{G_p}$ splits over the maximal compact subgroup $\mathrm{Sp}(2n,\mathbb{Z}_p)$ of $\mathrm{Sp}(2n,\mathbb{Q}_p)$. This allows us to define the restricted direct product

$$\mathop{\overset{\circ}{\times}}_p \widetilde{G_p}.$$

(We include $p = \infty$ in this product, where $\mathbb{Q}_\infty = \mathbb{R}$.) This is a central extension of $G_{\mathbb{A}}$:

$$(**) \qquad e \to \bigoplus_p \mu_p \to \mathop{\overset{\circ}{\times}}_p \widetilde{G_p} \to G_{\mathbb{A}} \to e,$$

where we let $\mu_p = \mathbb{Z}$ for $p = \infty$.

For each p, there is a unique surjection $\sigma_p \colon \mu_p \to \mu$, where $\mu \cong \mathbb{Z}/2\mathbb{Z}$ is the group $\{\pm 1\}$ of roots of unity in $\mathbb{Q}$. The sum of these homomorphisms is a homomorphism

$$\sigma \colon \bigoplus_p \mu_p \to \mu.$$

Applying σ to the kernel of the extension $(**)$ yields a central extension

$$e \to \mu \to E \to G_{\mathbb{A}} \to e.$$

Moore proved (not only for symplectic groups) that this extension splits over $G_{\mathbb{Q}}$, and that it is the universal extension with this property.

References

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