Foliation-preserving maps between solvmanifolds

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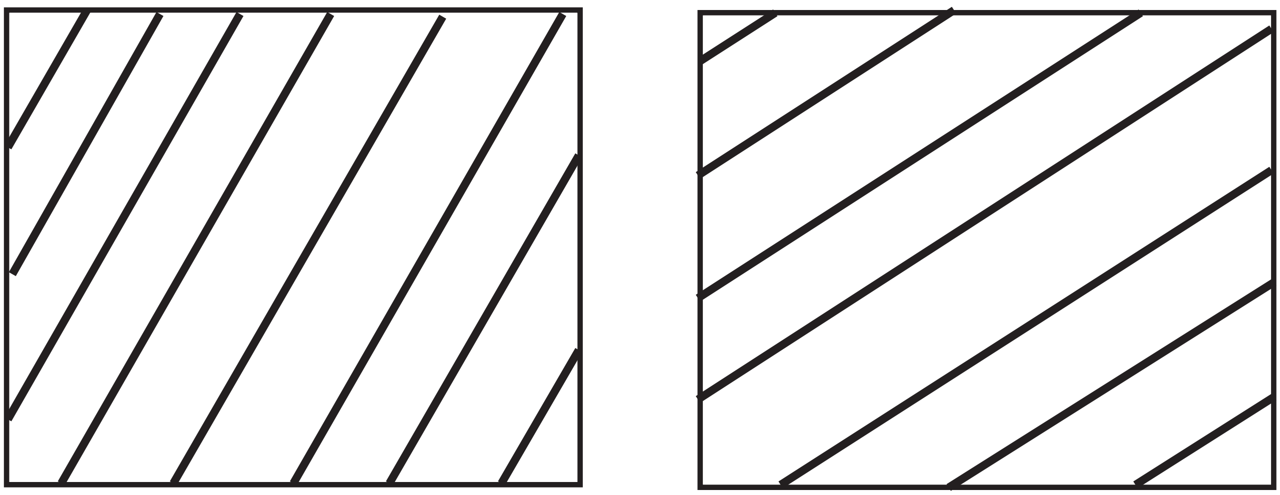
Eg. Let V_1, V_2 be connected Lie subgroups of $\mathbb{R}^n$.
Then V_i acts on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
The orbits of V_i define a foliation $\mathcal{F}_i$ of $\mathbb{T}^n$.



Question. *When is $\mathcal{F}_1$ topologically equiv to $\mathcal{F}_2$?*

Thm (Folklore). $\mathcal{F}_1 \sim \mathcal{F}_2$ *iff*
$$\sigma(V_1) = V_2 \text{ for some } \sigma \in \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^n).$$

Proof (D. Benardete). ($\Rightarrow$) Let f be a homeo of $\mathbb{T}^n$ that maps each V_1-orbit onto a V_2-orbit.

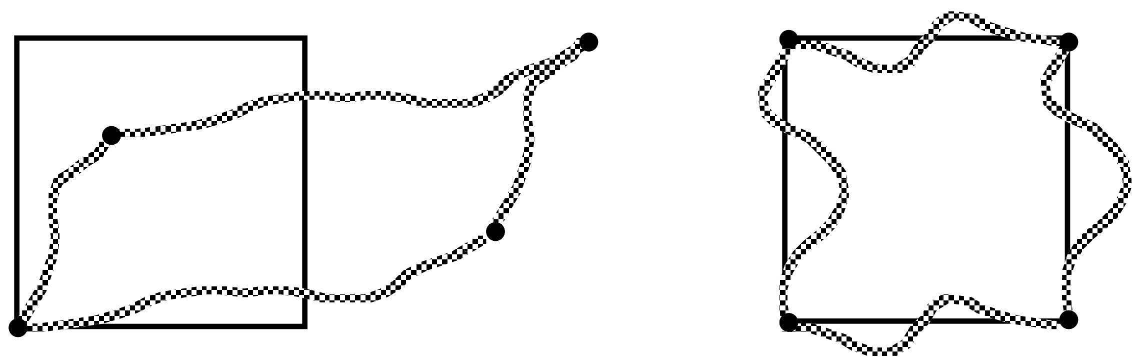
Lift f to a homeomorphism $\widetilde{f}$ of $\mathbb{R}^n$.

Compose $\widetilde{f}$ with a translation, so $\widetilde{f}(0) = 0$.

Then $\widetilde{f}|_{\mathbb{Z}^n} \in \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$, so

(*) $\widetilde{f}|_{\mathbb{Z}^n}$ extends to an automorphism σ of $\mathbb{R}^n$.

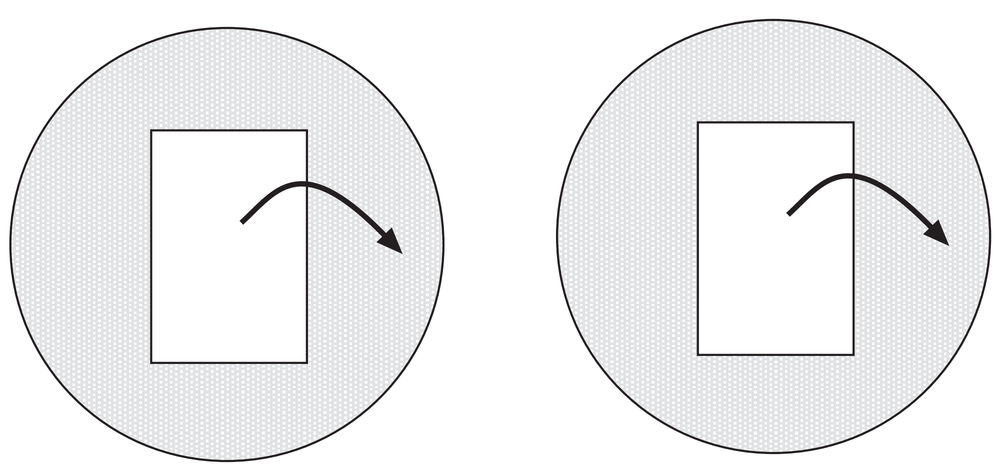
Compose $\widetilde{f}$ with σ^{-1}, so $\widetilde{f}|_{\mathbb{Z}^n} = \mathrm{Id}$.



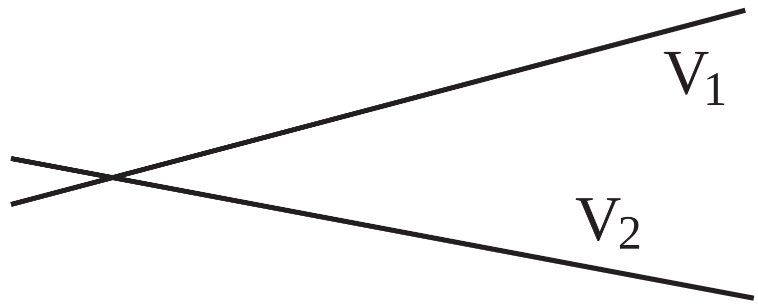
(**) $\mathbb{R}^n/\mathbb{Z}^n$ is compact,

so $\widetilde{f}$ moves points by a bounded amount,

i.e., $d(s, \widetilde{f}(s)) < C$ for all s.



Suppose $V_1 \neq V_2$. These are two subspaces of $\mathbb{R}^n$,



so it is obvious that

(***) there are elements of V_2 that are arbitrarily far from V_1 (or vice-versa).

We know $\widetilde{f}$ fixes 0 and maps V_1-orbits onto V_2-orbits. Thus $\widetilde{f}$ maps V_1 onto V_2.

But $\widetilde{f}$ moves points by only a bounded amount, so this is impossible. ■

More generally:

- G_i = simply conn solvable Lie group
- V_i = connected Lie subgroup of G_i
- Γ_i = lattice in G_i

The orbits of V_i define a foliation $\mathcal{F}_i$ of G_i/Γ_i.

$\mathcal{F}_1 \sim \mathcal{F}_2$ if $G_1 = G_2$, $V_1 = V_2$, and $\Gamma_1 = \Gamma_2$
(up to isomorphism).

Benardete's argument proves the converse if:

(*) every isomorphism of Γ_1 with Γ_2 extends to an isomorphism of G_1 with G_2;

(**) G_1/Γ_1 is compact; and

(***) if X and Y are two connected subgroups of G_1, and $X \not\subset Y$, then X diverges from Y.

Defn. *X diverges* from Y if $\nexists$ compact set $K \subset G$ with $X \subset YK$.

(**) G_i/Γ_i is compact.

OK: every lattice in a solvable Lie grp is cocpct

(***) if X and Y are two connected subgroups of G_1, and $X \not\subset Y$, then X diverges from Y.

Condition (***) can fail.

Eg. Let $G = \widetilde{\mathrm{SO}(2)} \ltimes \mathbb{R}^2$ and $Y = \widetilde{\mathrm{SO}(2)}$.

$$X = v^{-1} Y v \subset Y[Y, v] = Y (v^{-1})^Y v$$
$$= Y \cdot \big(\mathrm{SO}(2) v^{-1}\big) v = YK.$$

So X does not diverge from Y.

The trouble is: $\mathrm{Ad}G \cong \mathrm{SO}(2)$ is compact

(or: Zariski closure $\overline{\mathrm{Ad}G} \supset$ cpct torus)

(*) every isomorphism of Γ_1 with Γ_2 extends to an isomorphism of G_1 with G_2.

Condition (*) can fail.

Eg. Let $G_1 = \widetilde{\mathrm{SO}(2)} \ltimes \mathbb{R}^2$ and $G_2 = \mathbb{R}^3$.

$$\Gamma_1 = Z(G) \ltimes \mathbb{Z}^2 \cong \mathbb{Z}^3 = \Gamma_2$$

G_2 is abelian, but G_1 is not, so $G_1 \not\cong G_2$.

The trouble is: $\overline{\mathrm{Ad} G_1} \supset$ cpct torus

Thm (Benardete-Witte). $\overline{\mathrm{Ad} G_i} \not\supset$ *cpct torus. Then* $\mathcal{F}_1 \sim \mathcal{F}_2$ *iff* $\exists$ *iso* $\sigma \colon G_1 \to G_2$*, such that*

$$\sigma(V_1) = V_2 \text{ and } \sigma(\Gamma_1) \text{ is conjugate to } \Gamma_2.$$

Cor. $\overline{\mathrm{Ad} G_i} \not\supset$ *cpct torus.* $\mathcal{F}_1$ *has a dense leaf. If* $f \colon \mathcal{F}_1 \cong \mathcal{F}_2$*, then* $f = c \circ a$*, where*

- $c \colon \mathcal{F}_1 \to \mathcal{F}_2$ *is an affine map; and*
- $a \colon \mathcal{F}_1 \cong \mathcal{F}_1$ *maps each leaf onto itself.*

(Benardete and Witte assumed V_i unimodular.)

(*) every isomorphism of Γ_1 with Γ_2 extends to an isomorphism of G_1 with G_2.

I.e., every homeo $G_1/\Gamma_1 \sim G_2/\Gamma_2$ is homotopic to an affine map.

Thm (Witte). *Tech assumps on* Γ_1, Γ_2.
Any map $G_1/\Gamma_1 \to G_2/\Gamma_2$
lifts to a map $G_1/\Gamma_1 \to G_2^{\Delta}/\Gamma_2$
that is homotopic to an affine map.

Defn. Let T be a maximal compact torus of $\overline{\mathrm{Ad} G}$.
Then $G^{\Delta} = T \ltimes G \qquad \supset$ "nilshadow"

Thm (Bernstein-Witte). *Tech assump on* Γ_1, Γ_2.
If $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ *is a covering map on each leaf, then* $f = b \circ c_*'' \circ a$.

- $a\colon \mathcal{F}_1 \cong \mathcal{F}_1$ maps each leaf onto itself.
- $b\colon G_2/\Gamma_2 \to G_2/\Gamma_2$ is a translation.
- $c\colon G_1 \to G_2$ is a homomorphism.

We can usually modify a homo $c\colon G_1 \to G_2$.

Let $\delta\colon G_1 \to G_2^{\Delta}$ a homo with $\delta(\Gamma_1[G_1, G_1]) = e$.

Define $c'\colon G_1 \to G_2^{\Delta}$ by $c'(g) = \delta(g) \cdot c(g)$.

Under appropriate hypotheses,

- $G_2' = c'(G_1)$ is a conn Lie subgrp of G_2^{Δ};
- $V_2' = c'(V_1)$ is a subgroup of G_2'; and
- $\forall g \in G$, $c'|_{V_1 g}$ is homeo onto coset of V_2'.

Then $c'_*\colon \mathcal{F}_1 \to \mathcal{F}_2'$.

Can add δ' to get c''.

Rem. For $g, h \in G_1$, we have

$$\begin{aligned} c'(gh) &= \delta(gh)\ c(gh) \\ &= \delta(g)\ \delta(h)\ c(g)\ c(h) \\ &= \delta(g)\ c(g)^{\delta(h)^{-1}}\ \delta(h)\ c(h) \\ &= \big(\delta(g)\ c(g)\big)^{\delta(h)^{-1}}\ \delta(h)\ c(h) \\ &= c'(g)^{\delta(h)^{-1}}\ c'(h) \end{aligned}$$

So c' is a *crossed homomorphism.*
c'' is a *doubly-crossed homomorphism.*

Rem. If $[c(G_1), \delta(G_1)] \subset c(G_1 \cap \ker \delta)$, then

$$\begin{aligned} c'(g)^{\delta(h)} &= \delta(g)\ c(g)^{\delta(h)} \\ &= \delta(g)\ c(g)\ [c(g), \delta(h)] \\ &= \delta(g')\ c(g') \\ &= c'(g') \end{aligned}$$

so $c'(G_1)$ is a subgroup.

Rem. Our theorem holds in many cases where G_1 and G_2 are not solvable (and are not assumed to be semisimple, either), but our results are not definitive in the general case.

Technical assumptions.

- $\overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad}G_1}$
- $\overline{\mathrm{Ad}_{G_2}\Gamma'}$ is connected, $\forall \Gamma' \subset \Gamma_2$

(We use almost-Zariski closure here.)

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