Hamiltonian checkerboards

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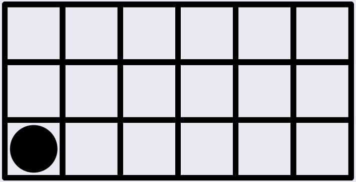
Abstract. Place a checker on some square of an $m \times n$ rectangular checkerboard. Asking whether the checker can tour the board, visiting all of the squares without repeats, is the same as asking whether a certain graph has a hamiltonian path. The question becomes more interesting if we allow the checker to step off the edge of the board. This topic has connections with ideas from elementary topology and group theory, and has been studied by Duluth REU students and other mathematicians for over 40 years. No advanced mathematical training will be needed to understand most of this talk.

slides at https://deductivepress.ca/dmorris/gallianfest.pdf

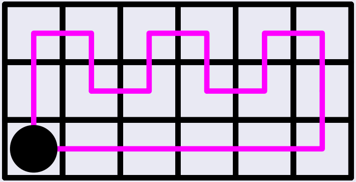
A checker is in the Southwest corner of an $m \times n$ checkerboard.

Can the checker tour the board?



- The checker (rook?) moves North, South, East, West (not diagonally!)
- A tour must visit each square exactly once *and return to the starting point* (hamiltonian cycle).

A checker is in the Southwest corner of an $m \times n$ checkerboard.



Can the checker tour the board?

Yes if mn is **even**. ✓

No if mn is **odd**.

Proof.

NORTH = SOUTH and EAST = WEST.

TOTAL = NORTH + SOUTH + EAST + WEST
= (2 × NORTH) + (2 × EAST)

is an even number.

TOTAL = # squares on the checkerboard = mn is odd. □

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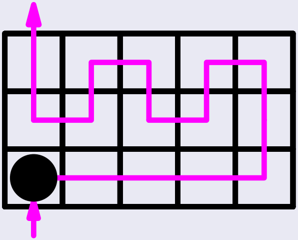
Can the checker tour the board?



Allow the checker to step off the edge of the board.

(The board is now toroidal, rather than flat.)

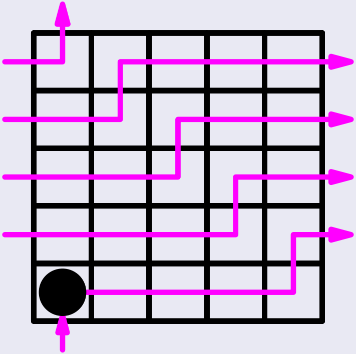
Proposition

A checker can tour any board if allowed to step off the edge.

Allow the checker to step off the edge of the board.

Find a route that always travels North or East.

This is easy on any square checkerboard.



Definition

A board is ***hamiltonian*** if it has such a tour.

Square checkerboards ($n \times n$) are *hamiltonian*:
they have a tour that travels only North and East.

Proposition

The $m \times n$ checkerboard is **not** *hamiltonian if* $\gcd(m, n) = 1$.

Proof by contradiction.

The tour must have mn steps: $E + N = mn$.

E is divisible by m.
N is divisible by n, so E is also divisible by n.

Therefore E is divisible by $\operatorname{lcm}(m, n) = mn$.
so E is either mn or 0.
So the tour either never leaves one row,
or never leaves one column. $\rightarrow\leftarrow$ $\square$

Theorem (R. A. Rankin 1948, Trotter-Erdös 1978)

The $m \times n$ checkerboard is hamiltonian if and only if $mn = E + N$ for some $E, N \in \mathbb{Z}^{\geq 0}$ with $\gcd\left(\frac{E}{m}, \frac{N}{n}\right) = 1$.

Proof (⇒, Stephen Curran 1985).

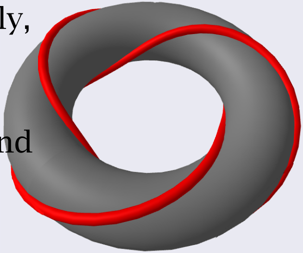
The board is a torus, so the path traced out by the checker is a closed path on the torus — a *torus knot*.

Let $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ be the knot class of this knot.

(The knot wraps x times longitudinally, and wraps y times meridionally.)

In other words, the checker steps off:

- the East edge of the board x times, and
- the North edge of the board y times.



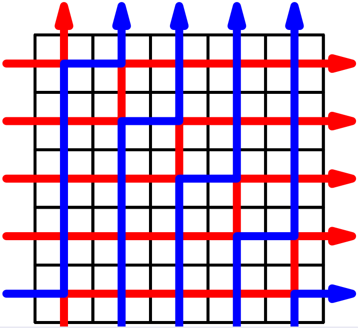
So $E = mx$ and $N = ny$.

Topology: since (x, y) is a knot class, $\gcd(x, y) = 1$. □

Theorem (R. A. Rankin 1948, Trotter-Erdös 1978)

The $m \times n$ checkerboard is hamiltonian if and only if $mn = E + N$ for some $E, N \in \mathbb{Z}^{\geq 0}$ with $\gcd\left(\frac{E}{m}, \frac{N}{n}\right) = 1$.

Exer. $\gcd(m,n) = a + b$ with $\gcd(a,m) = 1 = \gcd(b,n)$.



Corollary (John Lindgren, 1985)

The $m \times n$ checkerboard has two ***arc-disjoint*** *hamiltonian cycles*

$\Leftrightarrow \gcd(m,n) = a + b$ *with* $\gcd(ab, mn) = 1$.

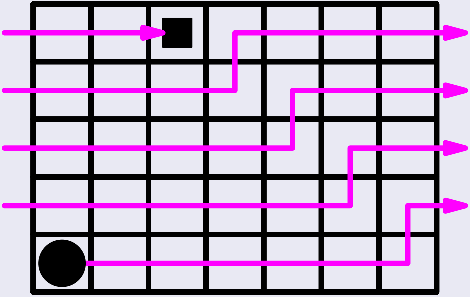
Remark. REU Duluthians looked at other cycles.
1980s: Doug Jungreis, Larry Penn, Amie Wilkinson, Joe Gallian.
2000s: Sherry Wu, Steve Curran,
Dan Isaksen's REU (Barone, Mauntel, Miller)

Change the rules: A tour must visit each square exactly once *but need not return to the starting place.* ("ham path")

Observation

Every checkerboard has a hamiltonian path.

Proof.



□

Where can hamiltonian paths end? (starting in SW corner)

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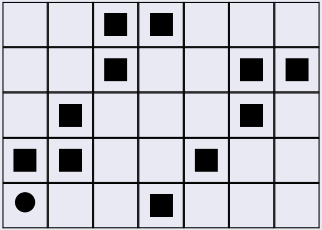
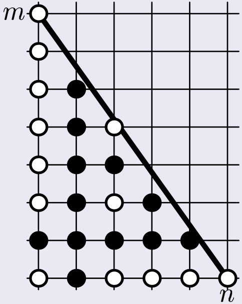
Example





For simplicity **assume $\mathbf{gcd}(m, n) = 1$.**

Recall. **lattice point**: $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. **primitive**: $\gcd(x, y) = 1$.

Corollary (Steve Curran 1985)

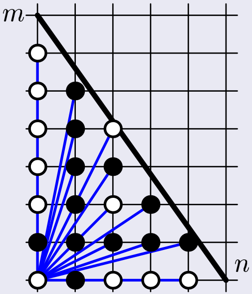
 # endpoints = # primitive lattice points in $\Delta(m, n)$ $- 1$.

$$\approx \frac{6}{\pi^2}\frac{mn}{2} \quad \approx 0.3mn \quad > mn/4$$

Where can hamiltonian paths end? (starting in SW corner)

Example.

34	19	**4**	**24**	9	29	14
13	33	**18**	3	23	**8**	**28**
27	**12**	32	17	2	**22**	7
6	**26**	11	31	**16**	1	21
20	5	25	**10**	30	15	0



Answer (Curran): **Rock-hopping at Split Rock in the dark**

$\Delta(m, n)$ is Lake Superior, and lattice points are rocks.

The lighthouse is at $(0, 0)$, and rotates counterclockwise.

Hop through all of the rocks (as efficiently as possible),

but can only hop when your rocks are in the light.

Start at $(n - 1, 0)$, and end at $(0, m - 1)$.

You get a prize (ham path) each time you visit the lighthouse.

Where can hamiltonian paths end? (starting in SW corner)

Example.

34	19	**4**	**24**	9	29	14
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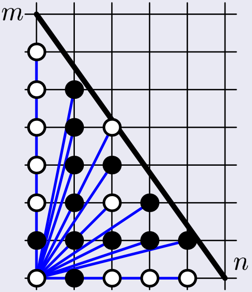

Theorem (Steve Curran 1985)

Answer is found by rock-hopping at Split Rock in the dark.

Proposition (David Housman 1981)

Hamiltonian path P_k ends at square k

$\Rightarrow N(P_k) = k$*: squares with smaller number travel north.*

Corollary (exercise). $N(P) \equiv N(P') \pmod 2$.

Question (Joe Gallian 1985 (personal communication))

When does the $m \times n$ checkerboard have two arc-disjoint hamiltonian **paths**?

Answer (Darijani-Miraftab-Morris 2023^+)

Always.

Proof when m and n are large

Need to choose two ham paths P_1 and P_2.

I.e., choose $N(P_1)$ and $N(P_2)$.

possibilities for $N(P_1), N(P_2) \approx \frac{3}{\pi^2} mn > \dfrac{mn}{4}$

$\dot{\Rightarrow}\ \exists\ N(P_1) + N(P_2) \in \{mn, mn-1\}$.

Translate P_2 to get

P_1: $\uparrow \uparrow \uparrow \uparrow \blacksquare \rightarrow \rightarrow$

P_2': $\blacksquare \rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow$

or P_2': $\blacksquare \rightarrow \rightarrow \rightarrow \rightarrow \uparrow \uparrow$

- J. A. Gallian and D. Witte: Hamiltonian checkerboards. *Math. Mag.* 57 (1984) 291–294.
- J. A. Gallian: Circuits in directed grids. *Math. Intelligencer* 13, no. 3 (1991) 40–43.
- S. J. Curran and D. Witte: Hamilton paths in Cartesian products of directed cycles. *Ann. Discrete Math.* 27 (1985) 35–74.
- K. Keating: Multiple-ply Hamiltonian graphs and digraphs. *Ann. Discrete Math.* 27 (1985) 81–87.
- D. Housman: Enumeration of Hamiltonian paths in Cayley diagrams. *Aequationes Math.* 23 (1981) 80–97.
- I. Darijani, B. Miraftab, and D. W. Morris: Arc-disjoint hamiltonian paths in Cartesian products of directed cycles (`arxiv:2203.11017`)

- V. Barone, M. Mauntel, and M. Miller: Hamiltonicity of the Cartesian product of two directed cycles minus a subgroup. *AKCE J. Graphs Combin.* 3 (2006) 39-43.
- S. X. Wu: Cycles in the Cartesian product of two directed cycles (unpublished).
- S. Curran, M. N. Ferencak, C. J. Morgana, and J. W. Thompson: The hamiltonicity of a Cayley digraph of a modified abelian group on two generators. Congressus Num. 188 (2007) 75–95.
- D. W. Morris: Hamiltonicity after reversing the directed edges at a vertex of a Cartesian product, JACODESMATH 10 (2023) 61–71.