Hamiltonian checkerboards

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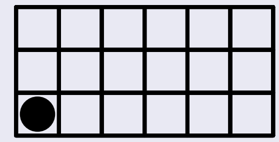
Abstract. Place a checker on some square of an $m \times n$ rectangular checkerboard. Asking whether the checker can tour the board, visiting all of the squares without repeats, is the same as asking whether a certain graph has a hamiltonian path (or hamiltonian cycle). The question becomes more interesting if we allow the checker to step off the edge of the board. This modification leads to numerous open problems, and also to connections with ideas from elementary topology and group theory. Some of the problems may be easy, but many have resisted attack for 30 years. No advanced mathematical training will be needed to understand most of this talk.

A checker is in the Southwest corner of an $m \times n$ checkerboard.

Can the checker tour the board?



- The checker (rook?) moves North, South, East, West (not diagonally!)
- A tour must visit each square exactly once *and return to the starting point* (hamiltonian cycle).

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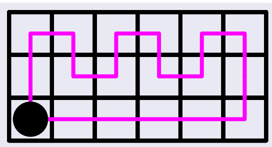

Yes if mn is **even**. ✓

No if mn is **odd**.

Proof.

NORTH = SOUTH and EAST = WEST.
TOTAL = NORTH + SOUTH + EAST + WEST
= (2 × NORTH) + (2 × EAST)
is an even number.
TOTAL = # squares on the checkerboard = mn is odd. □

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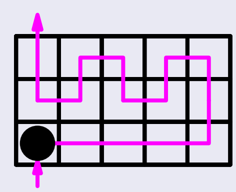
Can the checker tour the board?



Allow the checker to step off the edge of the board.
(The board is now toroidal, rather than flat.)

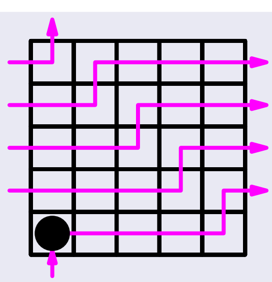
Proposition

A checker can tour any board if allowed to step off the edge.

Allow the checker to step off the edge of the board.

Find a route that always travels North or East.

This is easy on any square checkerboard.



Definition

A board is *hamiltonian* if it has such a tour.

Square checkerboards ($n \times n$) are *hamiltonian*:
they have a tour that travels only North and East.

Proposition

The $m \times n$ *checkerboard is* **not** *hamiltonian if* $\gcd(m, n) = 1$.

Proof by contradiction.

The tour must have mn steps: $E + N = mn$.
E is divisible by m.
N is divisible by n, so E is also divisible by n.
Therefore E is divisible by $\operatorname{lcm}(m, n) = mn$.
so E is either mn or 0.
So the tour either never leaves one row,
or never leaves one column. →← □

Theorem (R. A. Rankin, Trotter-Erdös, Curran)

The $m \times n$ checkerboard is hamiltonian if and only if $mx + ny = mn$ for some $x, y \in \mathbb{Z}^{\geq 0}$ with $\gcd(x, y) = 1$.

Proof ($\Rightarrow$, Stephen Curran).

The board is a torus, so the path traced out by the checker is a closed path on the torus — a *torus knot*.

Let $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ be the knot class of this knot.
(The knot wraps x times longitudinally, and wraps y times meridionally.)

In other words, the checker steps off:

- the East edge of the board x times, and
- the North edge of the board y times.

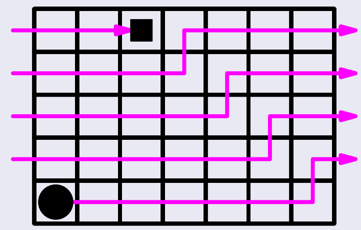
The tour has mx steps East, and ny steps North.
Therefore, $mx + ny = mn$.
Since (x, y) is a knot class, $\gcd(x, y) = 1$. □

Change the rules: A tour must visit each square exactly once but need not return to the starting place. ("ham path")

Observation

Every checkerboard has a hamiltonian path.

Proof.



□

Where can hamiltonian paths end? (starting in SW corner)

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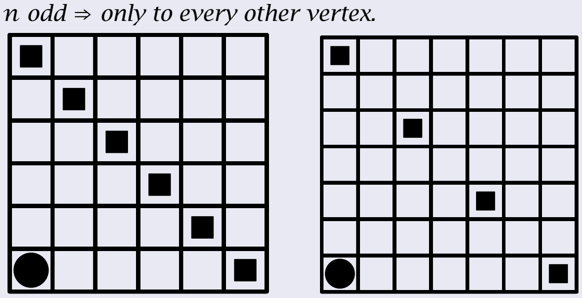
Theorem

On an $n \times n$ (square) checkerboard:

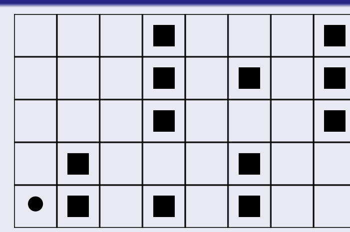
- *Hamiltonian paths always end on the main diagonal.*
- *n even $\Rightarrow$ $\exists$ ham path to anywhere on main diagonal.*
- *n odd $\Rightarrow$ only to every other vertex.*



Where can hamiltonian paths end? (starting in SW corner)

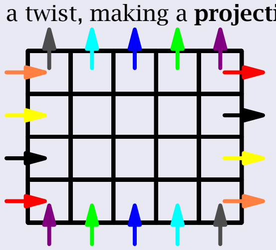
Hard problem in general, but solved by Stephen Curran (while he was an undergraduate)

Example



Change the topology

Instead of gluing the edges of the board to make a torus, we could glue with a twist, making a **projective plane**.



Proposition

No hamiltonian cycle (*unless m or n is* ≤ 2).

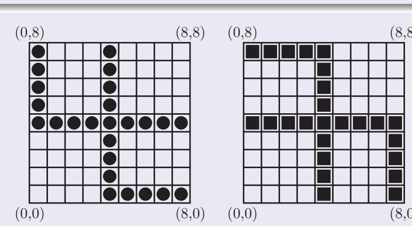
Corollary

No hamiltonian path starts in the Southwest corner.

Proposition

Only certain squares are the starting point of a ham path, and only certain squares are the ending point of a ham path.

Example:



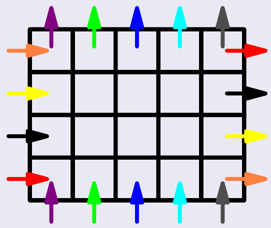
Problem

What squares can be the starting point of a ham path on an $m \times n$ projective checkerboard? (*Solved for $m = n$.*)

Remark: Starting points known $\Longrightarrow$ ending points known.

Another topology: Klein bottle

- glue top to bottom without a twist (like torus), and
- glue left to right with a twist (like projective plane).



Problem

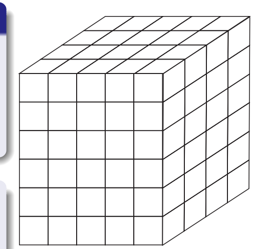
Find starting points and ending points of hamiltonian paths on a Klein checkerboard.

Even the square ($n \times n$) boards have not been studied.

Let's look at higher dimensions.

Exercise

Every 3-dimensional checkerboard has a hamiltonian path.
(North, East, Up)
Same for 4D, 5D, 6D, . . .



Hint: Each level of a 3D board can be thought of as a 2D board.
traverse all the cubes in a level,
then move up to the next level.

Theorem

Every 3-dimensional checkerboard has a hamiltonian cycle.
Same for 4D, 5D, 6D, . . .

Exercise

Every 3-dimensional checkerboard has a hamiltonian path
(starting in Southwest corner of bottom level).

What are the endpoints of ham paths in a 3D checkerboard?

Conjecture

If $\gcd(\ell, m, n) = 1$ *(and* $\ell, m, n \geq 2$*), then ham paths in the* $\ell \times m \times n$ *checkerboard can end anywhere.*

Remark

In $n \times n \times n$ cube, ham paths end anywhere on the "diagonal"
$\{ (x, y, z) \mid x + y + z \equiv -1 \pmod{n} \}$

Group theorist's point of view

Definition

- A = (finite) abelian group; e.g., $A = \mathbb{Z}_m \oplus \mathbb{Z}_n$,
- S = generating set; e.g., $S = \{(1,0), (0,1)\}$.

Cayley digraph $\mathrm{Cay}(A; S)$ is a digraph:

- vertices are elements of A,
- directed edge $v \to v + s$, for each $s \in S$.

Observation

- $\mathrm{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; \{(1,0), (0,1)\})$ *has a (directed) ham cycle*
  $\iff$ $m \times n$ *checkerboard is hamiltonian.*
- $\mathrm{Cay}(\mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n; \{(1,0,0), (0,1,0), (0,0,1)\})$
  has a (directed) ham cycle

Problem

For $\mathrm{Cay}(A; S)$ *(with* A *abelian):*

- *Is there a directed hamiltonian cycle?*
- *What are endpoints of directed hamiltonian paths?*

We know answers when $\#S \leq 2$.

For $\#S \geq 3$, problem is open even for cyclic group $A = \mathbb{Z}_n$.
Which circulant digraphs are hamiltonian?

Conjecture

$\mathrm{Cay}(A; S)$ *no directed ham cycle, and* $\#S \geq 3$ $\left(\begin{smallmatrix}\text{with } A \text{ abelian} \\ \text{and } e \notin S\end{smallmatrix}\right)$
$\implies$ $A = \mathbb{Z}_{2n}$ *is cyclic of even order, and* $\#S = 3$.

Remark: Detailed conjecture gives specific description of S and n.

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