Introduction to Arithmetic Groups

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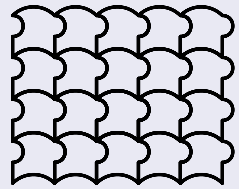
Abstract. We will discuss a few basic properties of "arithmetic groups," which are certain groups of $n \times n$ matrices with integer entries. By definition, the subject combines algebra (group theory and matrices) with number theory (the integers), but it also has connections with other areas, including the theory of periodic tilings. To learn more about these important groups, download a free copy of my book from http://arxiv.org/src/math/0106063/anc/

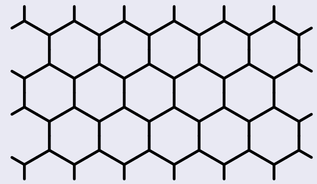
Group theory = the study of symmetry

Example

Symmetries of a **tessellation** (periodic tiling)



symmetry group $\Gamma = \mathbb{Z}^2$



$\Gamma \doteq \mathbb{Z}^2$

Thm (Bieberbach, 1910). $\forall$ tess of $\mathbb{R}^n$, $\Gamma \doteq \mathbb{Z}^n$.

Other spaces yield groups that are more interesting.

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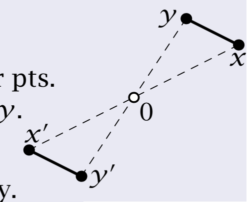
$\mathbb{R}^n$ is a **symmetric space**:

- **homogeneous**: every pt looks like all other pts. $\forall x, y$, $\exists$ isometry $x \mapsto y$. (preserves distances)
- reflection through a point $(x' = -x)$ is an isometry.

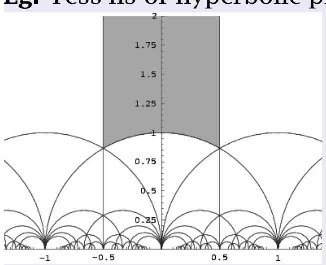


Example

- Sphere Γ = finite $\doteq \{e\}$
- Hyperbolic plane $\mathfrak{H}^2$ (upper half plane) (Poincaré disk)

drawn by KaleidoTil

Eg. Tess'ns of hyperbolic plane $\mathfrak{H}^2$. (symmetric space)



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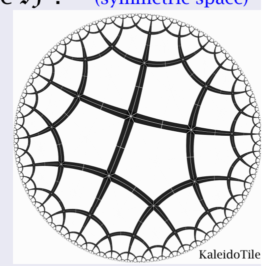


KaleidoTile

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$ $\quad \Gamma \subset \mathrm{Isom}(\mathfrak{H}^2) \doteq \mathrm{SL}(2,\mathbb{R})$

$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \,\middle|\, \det = 1 \right\}$.

My book: larger isometry groups
e.g., $\mathrm{SL}(n,\mathbb{R}) = \{\, g \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \det g = 1 \,\}$.

Tess'ns of symmetric spaces (e.g., $\mathbb{R}^n$, sphere, $\mathfrak{H}^2$).
My book: larger isometry groups (e.g., $\mathrm{SL}(n,\mathbb{R})$).

Theorem (classical, ≤ E. Cartan, 1926)

$\mathrm{SL}(n,\mathbb{R}) \doteq$ *isometry group of a symmetric space* $\mathcal{H}_n$.

My book: symmetry groups of tessellations of $\mathcal{H}_n$ and other non-Euclidean, non-compact symmetric spaces.

Theorem (A. Borel, 1963)

Every symmetric space has a tessellation (∞'ly many Γ).

Theorem (Margulis Arithmeticity Thm, ~1975)

List of all possibilities for Γ (*if* $n \geq 3$).
Γ *is grp of mats with* $\mathbb{Z}$ *entries* ("*arithmetic grp*")

Theorem (A. Borel, 1963)

Every symmetric space X *has a tessellation* (∞'ly many).

Lemma

X *has a tess* $\iff$ $G = \mathrm{Isom}(X)$ *has* ***lattice*** *subgrp* Γ:

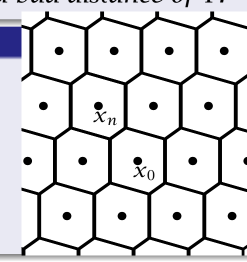
- Γ *is discrete* (*no acc pts*).
- *every el't of* G *is within a bdd distance of* Γ.

Proof.

($\Leftarrow$) Fix $x_0 \in X$.
For $x_n \in \Gamma x_0$ (discrete set),

$$T_n = \left\{ x \in X \,\middle|\, \begin{array}{l} x_n \text{ is closest} \\ \text{point of } \Gamma x_0 \end{array} \right\}.$$

This is a tessellation.
(*Voronoi diagram*) □



Thm (Borel). Every symm space has a tessellation.

Thm. $\mathrm{SL}(n,\mathbb{R})$ has a lattice subgroup.

Proof (n=2). **Quaternions** $\mathbb{H} = \{x + yi + zj + wk\}$.
$i^2 = j^2 = k^2 = -1$, $ij = k = -ji$

$\mathbb{H}_3 = \{$same, *except* $i^2 = 3 = k^2\} = \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ i = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix},\ j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ k = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}.$$

$\Gamma = \{\, g \in \mathbb{H}_3(\mathbb{Z}) \mid \det g = 1 \,\} \subset \mathrm{SL}(2,\mathbb{R})$.

- $\{\mathbb{Z}$-pts$\}$ of vector space has no acc pts $\Rightarrow \Gamma$ discrete.
- Every pt of v.s. is within bdd dist of a $\mathbb{Z}$-pt, $\overset{\text{approx}}{\Longrightarrow}$ every pt of G is within bdd dist of Γ.

Thm. $\mathbb{H}_3(\mathbb{Z})^\times$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ i = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix},\ j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ k = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}.$$

Observation

$\mathbb{H}_3(\mathbb{Q})$ *is a* ***division algebra*** (*every nonzero el't has inverse*).

Proof.

For $g = x + yi + zj + wk \in \mathbb{H}_3(\mathbb{Q})$,

$$\begin{aligned} \det g &= (x + \sqrt{3}y)(x - \sqrt{3}y) - (z + \sqrt{3}w)(-z + \sqrt{3}w) \\ &= x^2 - 3y^2 + z^2 - 3w^2 \qquad (= g\,\overline{g}) \\ &\neq 0 \quad (\text{for } x, y, z, w \in \mathbb{Q} \text{ (not all 0)}). \end{aligned}$$ □

Replace $\mathbb{H}_3(\mathbb{Q})$ with larger division alg D over $\mathbb{Q}$.
Then $D(\mathbb{Z})^\times$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$ (if $D(\mathbb{R}) \cong \mathrm{Mat}_{n\times n}(\mathbb{R})$).