Survey of invariant orders on arithmetic groups

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Abstract. At present, there are more questions than answers about the existence of an invariant order on an arithmetic group. We will discuss four different versions of the problem: the order may be required to be total, or allowed to be only partial, and the order may be required to be invariant under multiplication on both sides, or only on one side. One version is trivial, but the other three are related to interesting conjectures in the theory of arithmetic groups.

Γ = arithmetic group (or finitely generated group)

Eg. $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ = {3 × 3 integer matrices with det 1}
or $\mathrm{SL}(n,\mathbb{Z})$ or $\mathrm{SL}(n,\mathbb{Z}[\sqrt{2}])$

In general: Lie group G (connected) $\subset \mathrm{SL}(n,\mathbb{R})$
$\Gamma = G(\mathbb{Z}) = G \cap \mathrm{SL}(n,\mathbb{Z})$. (Assume technical conds.)

Question: ¿ $\exists$ invariant order $\prec$ on Γ ? (transitive, antisymmetric)

- *total* ($x \prec y$ or $x \succ y$ or $x = y$) or *partial*
- *left-invariant* ($x \prec y \implies ax \prec ay$, $\forall x, y, a$) or *bi-invariant* (also invariant on the right)

Not many answers yet (for Γ arithmetic).

Eg. $\mathbb{Z}$ has a bi-invariant total order (namely, $<$).

Left-invariant partial orders

Proposition

Γ has lots of left-invariant partial orders (unless torsion).

Proof.

Fix $g \in \Gamma$ (∞ order). Let $P = \{\, g^n \mid n > 0 \,\}$. (or other semigrp $\not\ni e$)

Define $x \prec y \iff x^{-1}y \in P$.

- transitive: $x \prec y$ & $y \prec z$
  $\implies x^{-1}z = (x^{-1}y)(y^{-1}z) \in P$
- left-invariant: $x \prec y$
  $\implies (ax)^{-1}(ay) = x^{-1}y \in P$. □

Bi-invariant total orders

Proposition

*Γ has **bi-invariant** total order $\implies \Gamma \twoheadrightarrow \mathbb{Z}$.* (if Γ f.g.)
*I.e., Γ is **indicable**.*

Cor. Every f.g. subgroup of Γ is indicable.
I.e., Γ is **locally** indicable.

Theorem (Kazhdan et al.)

Γ indicable arith grp $\implies G \doteq \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$.
(Group with Kazhdan's property (T) is not indicable.)

Cor. Usually no bi-invariant total order on Γ. (arith)

Exists on finite-index subgrp of every arith subgrp of $\mathrm{SO}(1,3)$.
(finite-index embeds in right-angled Artin grp [Agol, Wise], which has bi-invariant total order)

Bi-invariant partial orders

Recall: Usually no bi-invariant total order on Γ (arith).

I believe no (nontrivial) bi-invariant *partial* order unless $\mathrm{rank}_{\mathbb{R}} G = 1$
(i.e. $G = \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ or $\mathrm{Sp}(1,n)$ or $F_{4,1}$)

Equivalent:

- Every normal semigroup in Γ is a subgroup.
- $\forall g \in \Gamma$, e is a product of conjugates of g.

Known for G (and sometimes for $\mathbb{Q}$-points of G).

Problem: Prove for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$. (**US$100**)

I believe no (nontrivial) bi-invariant *partial* order unless $\mathrm{rank}_{\mathbb{R}} G = 1$

Theorem

$\mathrm{rank}_{\mathbb{R}} G = 1 \implies \Gamma$ *(relatively) hyperbolic*
$\implies \exists$ ***quasimorphism*** $\Gamma \to \mathbb{Z}$ [Epstein-Fujiwara]
$\implies \exists$ *normal semigroup that is not a subgroup*
$\implies \exists$ *bi-invariant partial order.*

Definition (quasimorphism)

$\varphi \colon \Gamma \to \mathbb{Z}$ (unbdd), $\varphi(\gamma_1) + \varphi(\gamma_2) - \varphi(\gamma_1\gamma_2)$ is bdd.

Exercise. Stabilize: $\overline{\varphi}(\gamma) = \lim \varphi(\gamma^n)/n$. Then:

- $\overline{\varphi}(\lambda^{-1}\gamma\lambda) = \overline{\varphi}(\gamma)$.
- $\{\, \gamma \in \Gamma \mid \overline{\varphi}(\gamma) > C \,\}$ is normal semigroup.

Left-invariant total orders

Recall. $\exists$ bi-inv't total order $\implies \Gamma$ locally indicable
$\implies G \doteq SO(1,n)$ or $SU(1,n)$ [Kazhdan]
$\implies \operatorname{rank}_{\mathbb{R}} G = 1.$ ($SO(1,n)$, $SU(1,n)$ $Sp(1,n)$, $F_{4,1}$.)

Proposition (Burns-Hale 1972)
Γ locally indicable $\implies \exists$ left-inv't total order.

Rough idea of proof.
For $x, y \in \Gamma$, $\exists \varphi \colon \langle x, y\rangle \twoheadrightarrow \mathbb{Z}$.
Define $x \prec y$ if $\varphi(x) < \varphi(y)$. □

Conjecture (1990's)
$\exists$ left-inv't total order on arith $\Gamma \implies$ $\operatorname{rank}_{\mathbb{R}} G = 1.$

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Theorem (Chernousov-Lifschitz-Morris, 2008)
If true for noncocompact in $SL(3,\mathbb{R})$ and $SL(3,\mathbb{C})$, then true for noncocompact in all G.

Noncocompact $\iff \exists$ subgrps that are *unipotent*
i.e., conjugate to subgroup of $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$.
Suffices to show: boundedly gen'd by unip subgrps.
$\exists$ unip subgrps $U_1, U_2, \ldots, U_n$, $\Gamma = U_1 U_2 \cdots U_n$.

Open question: ¿ $\exists$ *cocompact* arith group such that no finite-index subgrp has left-inv't total order ?

Conjecture (1990's)
$\exists$ left-inv't total order on arith $\Gamma \implies$ $\operatorname{rank}_{\mathbb{R}} G = 1.$
$G = SO(1,n)$ *or* $SU(1,n)$ *or* $Sp(1,n)$ *or* $F_{4,1}$.

$SO(1,3)$: every arith subgrp is left-orderable (up to finite index). [Agol: $\Gamma \twoheadrightarrow \mathbb{Z}$]
Maybe also $SO(1,n)$?

$SU(1,n)$: I don't know???

$Sp(1,n)$ and **$F_{4,1}$** have Kazhdan's property (T).
So probably no left-invariant total order.

Open question: ¿ $\exists$ left-orderable Kazhdan group ?

Summary

Γ = (irreducible) arithmetic group (in semisimple group G)
Assume $\operatorname{rank}_{\mathbb{R}} G \geq 2$.

Exercise
Γ has lots of left-invariant partial orders. (semigroups)

Proposition
Γ does not have a bi-invariant total order. ($\Gamma \not\twoheadrightarrow \mathbb{Z}$)

Conjecture
Γ has neither:
- *left-inv't total order* (*completely open for cocpct*), *nor*
- *bi-invariant partial order* (*completely open*).

Left-invariant total orders on lattices:

- D. W. Morris and Lucy Lifschitz: Bounded generation and lattices that cannot act on the line, Pure and Applied Mathematics Quarterly 4 (2008) 99–126. arxiv:math/0604612
- D. W. Morris: Some arithmetic groups that do not act on the circle (to appear). arxiv:1210.3671
- D. W. Morris: Can lattices in $SL(n,\mathbb{R})$ act on the circle? in B. Farb and D. Fisher, eds.: Geometry, Rigidity, and Group Actions. University of Chicago Press, Chicago, 2011. arxiv:0811.0051
- S. Boyer, D. Rolfsen, and B. Wiest: Orderable 3-manifold groups, Ann. Inst. Fourier, Grenoble 55 (2005) 243–288. arxiv:math/0211110

Bi-invariant partial orders on Γ:

- D. B. Epstein and K. Fujiwara: The second bounded cohomology of word-hyperbolic groups. Topology 36 (1997), no. 6, 1275–1289.

Bi-invariant partial orders on G:

- D. Witte: Products of similar matrices. Proc. Amer. Math. Soc. 126 (1998) 1005–1015.

$\dot\Gamma \twoheadrightarrow \mathbb{Z}$:

- A. Lubotzky: Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. Ann. of Math. (2) 144 (1996), no. 2, 441–452. MR1418904
- I. Agol: The Virtual Haken Conjecture. Doc. Math. 18 (2013) 1045–1087. MR3104553, `http://www.math.uni-bielefeld.de/documenta/vol-18/33.html`