Cocompact Lattices

Dave Morris

Dept of Math and Comp Sci
University of Lethbridge
Lethbridge, AB, T1K 3M4

`Dave.Morris@uleth.ca`

`http://people.uleth.ca/~dave.morris/`

Conj (Valette). $\Gamma =$ *cocpct lattice* $\Rightarrow \Gamma$ *has property RD.*

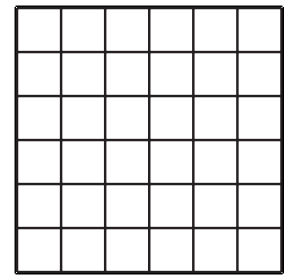
A simple example.

- $G = \mathbb{R}^2$ (connected Lie group)
- $\Gamma = \mathbb{Z}^2$ (discrete subgroup)

$G/\Gamma = \mathbb{T}^2$ is compact $\Rightarrow$ Γ is a cocpct lattice (in G).

$\Gamma = \pi_1(\mathbb{T}^2)$ = fund grp of locally symmetric space.

Fundamental domains tessellate $\mathbb{R}^2$.



$\Gamma \approx$ symmetries of tessellation of $\mathbb{R}^2$ (a symmetric space).

Replace $\mathbb{R}^2$ *with interesting* G.

Replace $\mathbb{R}^2$ *with interesting* G*:*
Lie group that is conn, noncpct, linear, simple (or ss).

- $\mathrm{SL}(n,\mathbb{R})$: $n \times n$ real mats of det 1
- $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$
- $\mathrm{SO}(m,n)$: isometries of $x_1^2 + x_2^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$
- $\mathrm{SU}(m,n)$: isometries of $|z_1|^2 + |z_2|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2$
- $\mathrm{Sp}(m,n)$: similar with $\mathbb{H}$
- etc.

Thm. $G = \mathrm{SL}(2,\mathbb{R})$ *or* $\mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$ *or* $\mathrm{Sp}(1,n)$
$\Rightarrow \Gamma$ *is Gromov hyperbolic* $\Rightarrow \Gamma$ *has RD.*

(*Henceforth, we assume* $\mathbb{R}$-rank $G \geq 2$.)

Thm. $G = \mathrm{SL}(3,\mathbb{R})$ *or* $\mathrm{SL}(3,\mathbb{C})$ *or* $\mathrm{SL}(3,\mathbb{H})$ *or* $\mathrm{SL}(3,\mathbb{O})$
$\Rightarrow \Gamma$ *has RD.*

Some small G's that are open:
$\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SO}(2,3) \approx \mathrm{Sp}(4,\mathbb{R})$.

Thm (Margulis Arithmeticity Theorem). Γ *is arithmetic.*

$$\approx \quad G \hookrightarrow \mathrm{SL}(n,\mathbb{R}),\ \Gamma = G \cap \mathrm{SL}(n,\mathbb{Z}).$$

- Γ is on an explicit list.

Cocompact lattices in $\mathrm{SO}(2,3)$**.**

Eg. Let

- $\alpha = \sqrt{2}$,
- $G = \mathrm{SO}(x_1^2 + x_2^2 - \alpha x_3^2 - \alpha x_4^2 - \alpha x_5^2) \cong \mathrm{SO}(2,3)$,
- $\Gamma = G_{\mathbb{Z}[\alpha]} = G \cap \mathrm{SL}(5, \mathbb{Z}[\alpha])$.

Then Γ is a cocompact lattice in G.

Idea of proof.

- $\sigma(a + b\alpha) = a - b\alpha$ (Galois auto of $\mathbb{Q}(\alpha)$),
- $G^\sigma = \mathrm{SO}(x_1^2 + x_2^2 + \alpha x_3^2 + \alpha x_4^2 + \alpha x_5^2) \cong \mathrm{SO}(5)$,

Map $\omega \mapsto (\omega, \omega^\sigma)$ embeds $\mathbb{Z}[\alpha] \hookrightarrow \mathbb{R} \oplus \mathbb{R}$.
Image is discrete. (Lattice in $\mathbb{R}^2$, so $\approx \mathbb{Z}^2$)
So image of Γ in $G \times G^\sigma$ is discrete (cocpct lattice!).

Can mod out compact group G^σ. $\square$

More general.

- $\alpha_1, \ldots, \alpha_5$ algebraic integers, s.t.
  - $\alpha_1, \alpha_2 > 0$ and $\alpha_3, \alpha_4, \alpha_5 < 0$,
  - $\forall$ Galois auts of $\mathbb{Q}(\alpha_1, \ldots, \alpha_5)$, $\alpha_i^\sigma > 0$.

Then

- $G = \mathrm{SO}(\alpha_1 x_1^2 + \cdots + \alpha_5 x_5^2) \cong \mathrm{SO}(2,3)$,
- $\Gamma = G \cap \mathrm{SL}(5, \mathbb{Z}[\alpha_1, \ldots, \alpha_5])$ is a cocpct lattice in G.

Special case of Margulis Arith Thm.
This constructs all cocompact lattices in $\mathrm{SO}(2,3)$.

Can describe arithmetic lattices in any G,
but answer may be more complicated.
(May need quaternion algebras — or other division algebras.)

Basic algebraic properties of cocompact lattices.

- Γ *is finitely generated* (in fact, finitely presented)
  (because $\Gamma \approx \pi_1(\text{cpct mfld})$)
- Γ *is linear* (a group of matrices) because G is linear
- Γ *is torsion free* (after pass to finite-index subgroup)
  (follows from preceding two [Selberg])
- Γ *is residually finite* ($\forall \gamma \in \Gamma, \exists \gamma \notin H \triangleleft \Gamma, \Gamma/H$ finite)
- Γ *contains free subgroup* (Tits Alternative)
  ($\Rightarrow$ exponential growth) ($\Rightarrow$ not amenable)

Fundamental algebraic properties.

- Γ has Kazhdan's Property T [Kazhdan et al.]
  (Γ acts on Hilbert space $\Rightarrow$ fixed pt)
- Γ is almost simple [Margulis]:
  $N \triangleleft \Gamma \Rightarrow \Gamma/N$ is finite (or $N = \{e\}$)
- **Conjecture** [Rapinchuk]: Γ *has bounded generation*
  i.e., product of finitely many cyclic subgroups:
  $\exists \gamma_1, \ldots, \gamma_r$, $\Gamma = \langle \gamma_1 \rangle \langle \gamma_2 \rangle \cdots \langle \gamma_r \rangle$
- **Conjecture** [Serre]: Γ *has Congruence Subgrp Prop*
  I.e., $N \triangleleft \Gamma \Rightarrow N \supset$ congruence subgroup.

Recall $\Gamma \approx G \cap \mathrm{SL}(n, \mathbb{Z})$ [Marg Arith Thm].
$N \supset \Gamma_m = \{\, \gamma \in \Gamma \mid \gamma \equiv \mathrm{Id} \pmod{m} \,\}$.

Connections with G.

- Γ is quasi-isometric to G

- Γ *is Zariski dense* [Borel Density Theorem]:
  - $\nexists$ connected $H < G$, s.t. $\Gamma \leq H$
  - $\rho: G \to \mathrm{SL}(n,\mathbb{C})$, $\rho(\Gamma)$ fixes v $\Rightarrow \rho(G)$ fixes v.
  - $\rho: G \to \mathrm{SL}(n,\mathbb{C})$, $\rho(\Gamma)$ fixes W $\Rightarrow \rho(G)$ fixes W.

- Γ *is superrigid* [Margulis]:

$\approx$ $\rho: \Gamma \to \mathrm{SL}(n,\mathbb{C})$ $\Rightarrow \rho$ extends to $\hat{\rho}: G \to \mathrm{SL}(n,\mathbb{C})$

  - pass to finite-index subgroup of Γ,
  - compose with Galois auto of $\mathbb{C}$,
  - tensor several of these together.

Consequences of superrigidity

- $\rho: \Gamma \to \mathrm{SL}(n,\mathbb{C})$ $\Rightarrow$
  - $\rho(\Gamma) \subset \mathrm{SL}(n, \text{algebraic numbers})$ (change basis)
  - $\rho(\Gamma) \subset \{\text{diagonalizable matrices}\}$

- *Margulis Arithmeticity Theorem*

Why superrigidity implies arithmeticity.

Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$, and assume H is superrigid.

We wish to show $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$,
i.e., want every matrix entry to be an integer.

First, let us show they are algebraic numbers.

Suppose some $\gamma_{i,j}$ is transcendental.
Then $\exists$ field auto φ of $\mathbb{C}$ with $\varphi(\gamma_{i,j}) = ???$

Define $$\tilde{\varphi}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{bmatrix}.$$

This map $\tilde{\varphi}: \Gamma \to \mathrm{GL}(n,\mathbb{C})$ is a group homo.

Superrigid: $\tilde{\varphi}$ extends to $\hat{\varphi}: \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{C})$.

There are uncountably many different φ's,
but $\mathrm{SL}(n,\mathbb{R})$ has only finitely many n-dim'l rep'ns $\to\leftarrow$

Now know every matrix entry is an algebraic number.

Second, show matrix entries are rational.

Fact. Γ is generated by finitely many matrices.

Matrix entries of these generators
generate a finite-degree field extension of $\mathbb{Q}$.
"algebraic number field"
So $\Gamma \subset \mathrm{SL}(n,F)$. For simplicity, assume $\Gamma \subset \mathrm{SL}(n,\mathbb{Q})$.

Third, show matrix entries have no denominators.
Actually, show denominators are bounded.
(Then finite-index subgrp has no denoms.)

Since Γ is generated by finitely many matrices,
only finitely many primes appear in denoms.

So suffices to show each prime only occurs to bdd power.

Γ is a superrigid lattice in $\mathrm{SL}(n, \mathbb{R})$
and every matrix entry is a rational number.

Need to show each prime only occurs to bounded power in denoms.

This is the conclusion of *p-adic superrigidity*:

Thm (Margulis).
If $\varphi\colon \Gamma \to \mathrm{SL}(\ell, \mathbb{Q}_p)$ is a group homomrphism, then $\varphi(\Gamma)$ has compact closure.
I.e., $\exists k$, no matrix in $\varphi(\Gamma)$ has p^k in denom.

Summary of proof:

1) $\mathbb{R}$-superrigidity $\Rightarrow$ matrix entries “rational”

2) $\mathbb{Q}_p$-superrigidity $\Rightarrow$ matrix entries $\in \mathbb{Z}$